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A NEW APPROACH TO THE NON-LINEAR PROBLEMS OF FM CIRCUITS*

BY

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Abstract. Closed form expressions are developed for the output of a frequency modulation receiver for an arbitrary number of superposed input signals. This corresponds to problems of interference or disturbance due to scatter and multiple reflexions. It is also shown how the Fourier components of the output may be evaluated by methods more direct than the usual Fourier analysis.

Introduction. A frequency modulation receiver is essentially a non-linear device. The input signal is usually fed into two non-linear filters, first into an amplitude limiter which reduces the signal to a constant amplitude, then into a discriminator whose output is a rectified signal with an amplitude proportional to the frequency deviation from the carrier frequency. Sometimes this output of the discriminator is processed through a linear filter which eliminates all but a few frequency components of the modulation.

Because of the non-linearity of the system, special methods must be devised to evaluate the output due to the superposition of input signals. The procedure presented here yields closed form expressions for the output when an arbitrary number of signals or a continuous distribution of them are superposed. The results may be used to predict interference effects or the disturbance due to multiple reflexions or scattering of the main signal. The method makes use of the concept of instantaneous frequency. The limitations of this concept in analyzing the behavior of frequency modulation circuits was discussed extensively by Carson and Fry[1] and Van der Pol[2].

The general theory is developed in Section 1 for the case of an arbitrary input represented by a continuous spectrum. This is applied in Section 2 to the case of an arbitrary number of signals with a single modulation frequency. It is also indicated how the method applies when there is more than one modulation frequency. Section 3 deals with the Fourier analysis of the output. Because of the fact that the expression for the output is in the form of the quotient of two Fourier series, methods more direct than the usual Fourier analysis are applicable. It is shown that the Fourier coefficients

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[1] J. R. Carson and T. C. Fry, *Variable frequency electric circuit theory with application to the theory of frequency modulation*, Bell Syst. Tech. Journal **26**, 513-540 (1937).

[2] B. Van der Pol, *Fundamental principles of frequency modulation*, Journal I.E.E. **111**, 153-158 (1946).

may be evaluated directly by use of the theorem of residues of analytic functions. The method yields directly the Fourier components of the output in both amplitude and phase for any number of superposed input signal. In particular, the input may be represented by the superposition of its spectral components. This has an important bearing on scatter problems since it is then reduced to the calculation of the scatter for each monochromatic component. The procedure has been applied to a number of practical cases and found to be quite satisfactory from the standpoint of simplicity and accuracy. These applications along with some further simplifications will be presented in a subsequent paper.

1. General expressions for the output signal of an FM receiver. We consider an input signal which is both amplitude and frequency modulated,

$$E(t) = \tfrac{1}{2}I(t)[\exp(i\phi) + \exp(-i\phi)] \tag{1.1}$$

with

$$\phi = \omega_c t + \varphi(t),$$

where ω_c represents the carrier frequency, $\varphi(t)$ the frequency modulation and $I(t)$ the amplitude modulation. In the receiver the amplitude is first reduced to a constant value in a limiter circuit. It is then processed through a discriminator. This is usually made of circuits slightly off resonance with the carrier frequency, such that the output is a rectified voltage proportional to the frequency deviation of the signal from the carrier frequency. This frequency deviation being $d\phi/dt - \omega_c$, the ouput of the receiver is then given by

$$K\left(\frac{d\phi}{dt} - \omega_c\right) = K\,\frac{d\varphi}{dt}. \tag{1.2}$$

Sometimes this output signal is passed through a linear filter so as to extract frequency components in a narrow band.

In practice, the signal is not always given in the form (1.1) so that one cannot use (1.2) to evaluate the receiver output. In particular, we are interested in the calculation of the receiver output when the incoming signal is given by its spectrum

$$E(t) = \int_{-\infty}^{+\infty} G(\omega)e^{i\omega t}\,d\omega. \tag{1.3}$$

We establish a mathematical processing of the expression (1.3) by which it is possible to evaluate a quantity proportional to the rate of change of the phase angle ϕ, hence also proportional to the receiver output. To do this we apply to the signal a linear operator which consists in replacing its spectrum $G(\omega)$ by

$$G(\omega)(1 + K\Omega), \tag{1.4}$$

where

$$\Omega = |\,\omega\,| - \omega_c{}^*$$

is the frequency deviation of the spectral component from the carrier frequency. The effect of this linear operation on the signal can be readily evaluated in the form (1.3)

*If we wanted to restrict ourselves to analytic functions, the same purpose could be accomplished by putting $\Omega = (\omega^2 - \omega_c^2)/2\omega_c$.

by multiplying $G(\omega)$ by $1 + K\Omega$ in the integral. We shall now introduce the basic assumption of the method, namely that this operation is approximately equivalent to multiplying the signal by the factor

$$A = 1 + K\frac{d\varphi}{dt}. \tag{1.5}$$

This assumption is essentially the same as that upon which is based the design of a discriminator circuit, namely that the response of the circuit to the instantaneous amplitude and frequency of the signal is the same as in a steady state. The assumption will, of course, apply if the frequencies at which the amplitude, I, and phase angle, φ, vary are very much smaller than the carrier frequency, ω_c.
If we consider the signal in the form (1.1), it becomes

$$E_1(t) = \tfrac{1}{2}AI(t)[\exp(i\phi) + \exp(-i\phi)]. \tag{1.6}$$

Squaring this quantity we obtain

$$E_1^2(t) = \tfrac{1}{2}A^2I^2 + \tfrac{1}{4}A^2I^2[\exp(2i\phi) + \exp(-2i\phi)]. \tag{1.7}$$

We notice that the first term represents the low frequency components, while the second term represents the high frequency components. In practice, these components are widely separated. We may write

$$\mathcal{L}E_1^2(t) = \tfrac{1}{2}A^2I^2, \tag{1.8}$$

where the symbol $\mathcal{L}$ signifies "low frequency part of $\cdots$". Similarly, if we square the original signal $E(t)$ we may write

$$\mathcal{L}E^2(t) = \tfrac{1}{2}I^2. \tag{1.9}$$

Hence,

$$\frac{\mathcal{L}E_1^2(t)}{\mathcal{L}E^2(t)} = A^2 = 1 + 2K\frac{d\varphi}{dt} + K^2\left(\frac{d\varphi}{dt}\right)^2. \tag{1.10}$$

The linear term in K in the expression is proportional to the output signal of the receiver. We may write this output signal as

$$M(t) = K\frac{d\varphi}{dt} = \frac{K}{2\mathcal{L}E^2(t)}\left[\frac{d}{dK}\mathcal{L}E_1^2(t)\right]_{K=0}. \tag{1.11}$$

This expression will now be evaluated in terms of the representation (1.3) of the input by means of a spectrum. The square of the signal is the double integral.

$$E^2(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(\xi)G(\omega)e^{i(\xi+\omega)t}\,d\xi\,d\omega. \tag{1.12}$$

The integral is extended to the infinite plane. We introduce the following change of variables

$$\xi = \tfrac{1}{2}(\eta + \zeta), \qquad \omega = \tfrac{1}{2}(\eta - \zeta) \tag{1.13}$$

and derive

$$E^2(t) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G\left(\frac{\eta+\zeta}{2}\right)G\left(\frac{\eta-\zeta}{2}\right)e^{i\eta t}\,d\eta\,d\zeta. \tag{1.14}$$

The spectrum of $2E^2(t)$ is therefore

$$R(\eta) = \int_{-\infty}^{+\infty} G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) d\zeta. \tag{1.15}$$

In evaluating this expression we take into account the fact that $G(\omega)$ is small except in the vicinity of $\omega = \pm \omega_c$. Hence, the contribution to the integral will be only in the vicinity of the four points (see Fig. 1).

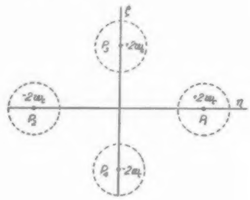

FIG. 1.

$$\begin{array}{lll} P_1: & \eta = 2\omega_c & \zeta = 0 \\ P_2: & \eta = -2\omega_c & \zeta = 0 \\ P_3: & \eta = 0 & \zeta = 2\omega_c \\ P_4: & \eta = 0 & \zeta = -2\omega_c \end{array}$$

The low frequency components of $E^2(t)$ are therefore given by integrating (1.14) in the vicinity of points P_3 and P_4

$$\mathcal{L}E^2(t) = \frac{1}{2} \int_{P_3, P_4} G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) e^{i\eta t} \, dS, \tag{1.16}$$

the integral being evaluated over elements of area dS in the vicinity of P_3 and P_4. Similarly, the low frequency components of $E_1^2(t)$ are given by

$$\mathcal{L}E_1^2(t) = \frac{1}{2} \int_{P_3, P_4} [1 + K(|\omega| - \omega_c)][1 + K(|\xi| - \omega_c)] G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) e^{i\eta t} \, dS. \tag{1.17}$$

Hence,

$$\begin{aligned} &\frac{1}{2} K \left[\frac{d}{dK} \mathcal{L}E_1^2(t) \right]_{K=0} \\ &\quad = \frac{1}{2} K \int_{P_3, P_4} \left(\frac{|\eta + \zeta|}{4} + \frac{|\eta - \zeta|}{4} - \omega_c \right) G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) e^{i\eta t} \, dS. \end{aligned} \tag{1.18}$$

From (1.16), (1.18), and (1.11) we derive the receiver output

$$\begin{aligned} M(t) = K \int_{P_3, P_4} &\left(\frac{|\eta + \zeta|}{4} + \frac{|\eta - \zeta|}{4} - \omega_c \right) G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) e^{i\eta t} \, dS \\ &\cdot \left\{ \int_{P_3, P_4} G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) e^{i\eta t} \, dS \right\}^{-1}. \end{aligned} \tag{1.19}$$

We have thus expressed this output in terms of the input spectrum G. It is seen that, as required by the problem, this expression is independent of the input amplitude. Expression (1.19) may be put in a somewhat different form by introducing

$$\begin{aligned} F(\eta) &= \int_{-\infty}^{+\infty} \left(\frac{|\eta + \zeta|}{4} + \frac{|\eta - \zeta|}{4} - \omega_c \right) G\left(\frac{\eta + \zeta}{2}\right)\left(\frac{\eta - \zeta}{2}\right) d\zeta \\ &= \int_{-\infty}^{+\infty} \left(\frac{|\eta - \zeta|}{2} - \omega_c \right) G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) d\zeta. \end{aligned} \tag{1.20}$$

The equivalence of these two expressions is easily seen if we replace the variable ζ by $-\zeta$. This function constitutes the spectrum of the numerator of expression (1.19). With the spectrum $R(\eta)$ of the denominator, as defined by (1.15), we may write

$$M(t) = K \int_{-\epsilon}^{\epsilon} F(\eta) e^{i\eta t}\, d\eta \left\{ \int_{-\epsilon}^{\epsilon} R(\eta) e^{i\eta t}\, d\eta \right\}^{-1}. \tag{1.21}$$

The integrands in the expressions for $F(\eta)$ and $R(\eta)$ are different from zero only in the vicinity of $\zeta = \pm 2\omega_c$ and the variable η is restricted to the vicinity of the origin. The range of integration $-\epsilon < \eta < +\epsilon$ is that of the low frequency portion of the spectral functions $F(\eta)$ and $R(\eta)$. According to the footnote remark ([1]), we could also write for $F(\eta)$

$$F(\eta) = \int_{-\infty}^{\infty} \frac{1}{2\omega_c} \left[\frac{1}{4} (\eta - \zeta)^2 - \omega_c^2 \right] G\left(\frac{\eta + \zeta}{2}\right) G\left(\frac{\eta - \zeta}{2}\right) d\zeta. \tag{1.22}$$

This expression will be approximately equal to (1.20).

2. Application to the superposition of sinusoidally modulated signals. In certain cases the input signal is made up of the superposition of signals whose frequency modulation is sinusoidal with a common carrier frequency, but with a different phase for each modulation. The signal is then expressed as

$$\begin{aligned} 2E(t) = \sum_j R_j \exp \left[i\omega_c t + i\varphi_j + i\left(\frac{\Delta\omega}{\omega_1}\right) \sin (\omega_1 t + \psi_j) \right] \\ + \sum_j R_j^* \exp \left[-i\omega_c t - i\varphi_j - i\left(\frac{\Delta\omega}{\omega_1}\right) \sin (\omega_1 t + \psi_j) \right], \end{aligned} \tag{2.1}$$

where R_j and R_j^* are complex conjugates. Such a signal occurs, for instance, in the case of multiple reflection or scatter. In this case the phase differences are

$$\begin{aligned} \varphi_j &= -\omega_c t_j , \\ \psi_j &= -\omega_1 t_j , \end{aligned} \tag{2.2}$$

where t_j is the time lag for arrival of the jth component in the receiver. In order to apply the results of the previous section, we must represent the signal by its spectrum. We make use of the identity

$$\exp (i\beta \sin \theta) = \sum_{n=-\infty}^{\infty} J_n(\beta) \exp (ni\theta), \tag{2.3}$$

where J_n is the Bessel function of the first kind of order n. By putting β equal to the modulation index

$$\beta = \frac{\Delta\omega}{\omega_1} \tag{2.4}$$

and

$$\begin{aligned} B_n &= \textstyle\sum_i R_i \exp(i\varphi_i + in\psi_i) \\ B_n^* &= \textstyle\sum_i R_i^* \exp(-i\varphi_i - in\psi_i) \end{aligned} \tag{2.5}$$

we may write the signal as

$$2E(t) = \sum_{n=-\infty}^{\infty} B_n J_n \exp(i\omega_c t + in\omega_1 t) + \sum_{n=-\infty}^{\infty} B_n^* J_n \exp(-i\omega_c t - in\omega_1 t). \tag{2.6}$$

This latter expression constitutes the expansion of the signal into a discrete spectrum of equidistant frequencies. The integration (1.3) is here replaced by a summation. The low frequency components of the square of the signal corresponding to expressions (1.16) are given by

$$\begin{aligned} 4\mathfrak{L}E^2(t) = &\sum_{n=-\infty}^{+\infty} B_n B_n^* J_n^2 \\ &+ \exp(i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n-1}^* J_n J_{n-1} + \exp(-i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n+1}^* J_n J_{n+1} \\ &+ \exp(2i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n-2}^* J_n J_{n-2} + \exp(-2i\omega_1 t) \sum_{n=-\infty}^{+\infty} B_n B_{n+2}^* J_n J_{n+2} \\ &+ \text{etc.} \end{aligned} \tag{2.7}$$

We note that

$$\begin{gathered} \sum_{n=-\infty}^{+\infty} B_n B_{n+1}^* J_n J_{n+1} = \sum_{n=-\infty}^{+\infty} B_{n+1} B_n^* J_{n-1} J_n \\ \sum_{n=-\infty}^{+\infty} B_n B_{n+2}^* J_n J_{n+2} = \sum_{n=-\infty}^{+\infty} B_{n-2} B_n^* J_{n-2} J_n \\ \text{etc.} \end{gathered} \tag{2.8}$$

and put

$$\begin{aligned} C_k &= \sum_{n=-\infty}^{+\infty} B_n B_{n-k}^* J_n J_{n-k} \\ C_k^* &= \sum_{n=-\infty}^{+\infty} B_n^* B_{n-k} J_n J_{n-k} . \end{aligned} \tag{2.9}$$

Then (2.7) may be written

$$4\mathfrak{L}E^2(t) = C_0 + \sum_{k=1}^{+\infty} [C_k e^{ik\omega_1 t} + C_k^* e^{-ik\omega_1 t}]. \tag{2.10}$$

We must also evaluate $E_1(t)$. This is obtained by multiplying by $1 + K\Omega$ each frequency component in the expansion (2.6) of $E(t)$. In this case

$$\begin{aligned} \Omega &= |\,\omega\,| - \omega_c \\ \omega &= \pm(\omega_c + n\omega_1). \end{aligned} \tag{2.11}$$

We note that in practice the terms in the series (2.6) are vanishingly small for $|n| > \beta$ because, in that case, $J_n \cong 0$. It is therefore legitimate to write

$$\Omega = n\omega_1 . \tag{2.12}$$

Putting

$$A_n = 1 + nK\omega_1 \tag{2.13}$$

we find

$$2E_1(t) = \sum_{n=-\infty}^{+\infty} A_n B_n J_n \exp(i\omega_c t + in\omega_1 t) + \sum_{n=-\infty}^{+\infty} A_n B_n^* J_n \exp(-i\omega_c t - in\omega_1 t). \tag{2.14}$$

Proceeding as before

$$4\mathcal{L}E_1^2(t) = F_0 + \sum_{k=1}^{+\infty} [F_k \exp(ik\omega_1 t) + F_k^* \exp(-ik\omega_1 t)] \tag{2.15}$$

with

$$\begin{aligned} F_k &= \sum_{n=-\infty}^{+\infty} A_n A_{n-k} B_n B_{n-k}^* J_n J_{n-k} \\ F_k^* &= \sum_{n=-\infty}^{+\infty} A_n A_{n-k} B_n^* B_{n-k} J_n J_{n-k} . \end{aligned} \tag{2.16}$$

In order to obtain $d/dK\ \mathcal{L}E_1^2(t)$ we consider the factor A_nA_{n-k} which is the only one to contain K. We have

$$A_n A_{n-k} = [1 - nK\omega_1][1 + (n-k)K\omega_1] \tag{2.17}$$

and

$$\left[\frac{d}{dK} A_n A_{n-k}\right]_{K=0} = (2n-k)\omega_1 . \tag{2.18}$$

Hence,

$$4\left[\frac{d}{dK}\mathcal{L}E_1^2(t)\right]_{K=0} = \omega_1 H_0 + \omega_1 \sum_{k=1}^{+\infty} [H_k \exp(ik\omega_1 t) + H_k^* \exp(-ik\omega_1 t)] \tag{2.19}$$

with

$$\begin{aligned} H_k &= \sum_{n=-\infty}^{+\infty} (2n-k) B_n B_{n-k}^* J_n J_{n-k} \\ H_k^* &= \sum_{n=-\infty}^{+\infty} (2n-k) B_n^* B_{n-k} J_n J_{n-k} . \end{aligned} \tag{2.20}$$

The output signal due to the input $E(t)$ given by (2.1) is

$$\begin{aligned} M(t) &= \frac{K}{2\mathcal{L}E^2(t)}\left[\frac{d}{dK}\mathcal{L}E_1^2(t)\right]_{K=0} \\ &= \frac{K\omega_1}{2}\,\frac{H_0 + \sum_{k=-\infty}^{+\infty} [H_k \exp(ik\omega_1 t) + H_k^* \exp(-i\omega_1 kt)]}{C_0 + \sum_{k=-\infty}^{+\infty} [C_k \exp(ik\omega_1 t) + C_k^* \exp(-i\omega_1 kt)]} . \end{aligned} \tag{2.21}$$

The same method may be applied if the superposed signals are not modulated by a single sinusoidal component. Consider, for instance, a signal component $E_i(t)$ containing two simultaneous modulation frequencies ω_1 and ω_2 .

$$E_i(t) = \exp [i\omega_c t + i\varphi_1 + i\varphi_2] + \exp [-i\omega_c t - i\varphi_1 - i\varphi_2] \tag{2.22}$$

with

$$\begin{aligned} \varphi_1 &= \beta_1 \sin \omega_1 t \\ \varphi_2 &= \beta_2 \sin \omega_2 t. \end{aligned} \tag{2.23}$$

Using the identity (2.3) the spectrum of $E_i(t)$ is obtained by writing

$$\begin{aligned} E_i(t) &= \exp i\omega_c t \exp i\varphi_1 \exp i\varphi_2 \\ &= \exp i\omega_c t\left[\sum_{n=+\infty}^{+\infty} J_n(\beta_1) \exp (in\omega_1 t)\right]\left[\sum_{n=-\infty}^{+\infty} J_n(\beta_2) \exp (in\omega_2 t)\right]. \end{aligned} \tag{2.24}$$

Performing the multiplication yields a discrete spectrum. However, this time the frequency intervals are not equal. From (2.24) we derive the spectrum due to the superposition of signals of the type (2.22) with individual time lags and amplitude factors as in (2.1)

$$E(t) = \sum_i R_i E_i(t - t_i). \tag{2.25}$$

Proceeding as above, the spectrum may be used to evaluate the receiver output due to this superposition.

3. Fourier analysis of the receiver output. We consider the case of the superposition of input signals modulated by the same modulation frequency ω_1 . We have shown that the output is given by expression (2.21) which is the quotient of two Fourier series. It is, itself, a periodic function of frequency ω_1 , which we may expand into a Fourier series. We omit the constant factor in expression (2.21) and write, putting $\omega_1 t = \tau$

$$\begin{aligned} M(t) &= \frac{H_0 + \sum_{k=1}^{+\infty} [H_k \exp (ik\tau) + H_k^* \exp (-ik\tau)]}{C_0 + \sum_{k=1}^{+\infty} [C_k \exp (ik\tau) + C_k^* \exp (-ik\tau)]} \\ &= M_0 + \sum_{k=1}^{+\infty} [M_k \exp (ik\tau) + M_k^* \exp (-ik\tau)]. \end{aligned} \tag{3.1}$$

If the receiver output is filtered through a lowpass filter so that only the fundamental component of the Fourier series of M is observed, we must evaluate the coefficients M_1 and M_1^* . We have

$$\begin{aligned} 2\pi M_1 &= \int_0^{2\pi} M(\tau) \exp (-i\tau)\, d\tau \\ 2\pi M_1^* &= \int_0^{2\pi} M(\tau) \exp (i\tau)\, d\tau. \end{aligned} \tag{3.2}$$

The integrals may be evaluated by a method which takes advantage of the particular form of the function $M(t)$ in the present case. Consider the second integral and change the variable of integration τ to a complex variable p

$$p = \exp (i\tau) \tag{3.3}$$

and express $M(\tau)$ in terms of p

$$M(\tau) = \frac{H_0 + \sum_{k=1}^{+\infty} [H_k p^k + H_k^* p^{-k}]}{C_0 + \sum_{k=1}^{+\infty} [C_k p^k + C_k^* p^{-k}]}. \tag{3.4}$$

The value of M_1^* is then given by the contour integral on the unit circle

$$M_1^* = \frac{1}{2\pi i} \oint \frac{H_0 + \sum_{k=1}^{+\infty} [H_k p^k + H_k^* p^{-k}]}{C_0 + \sum_{k=1}^{+\infty} [C_k p^k + C_k^* p^{-k}]} \, dp. \tag{3.5}$$

The value of this integral is equal to the sum of the residues and depends on the poles contained within the unit circle. There are multiple poles at $p = 0$ and poles corresponding to the roots of the denominator. These are the roots of the equation

$$C_0 + \sum_{k=1}^{\infty} [C_k p^k + C_k^* p^{-k}] = 0. \tag{3.6}$$

If

$$p_1 = r_1 \exp (i\theta_1) \tag{3.7}$$

is a root of this equation, then

$$C_0 + \sum_{k=1}^{+\infty} [C_k r_1^k \exp (ik\theta_1) + C_k^* r_1^{-k} \exp (-ik\theta_1)] = 0. \tag{3.8}$$

The conjugate of this expression must also be zero, hence,

$$C_0 + \sum_{k=1}^{+\infty} [C_k^* r_1^k \exp (-ik\theta_1) + C_k r_1^{-k} \exp (ik\theta_1)] = 0. \tag{3.9}$$

But this expresses that

$$p_2 = \frac{1}{r_1} \exp (i\theta_1) \tag{3.10}$$

is also a root of equation (3.6). Therefore, the roots are grouped in pairs of the same argument θ_1 and such that the product of their moduli is unity. Half the roots will be inside the unit circle and to each of these roots corresponds an outside root on the same radius from the origin. If there are roots on the circle, the above conclusion does not hold. However, since the expression (3.6) on the circle represents the square of the amplitude of the input signal $I^2(t)$, this can only happen if the input vanishes. As an example, let us consider the case when the denominator contains only C_0 and C_1. This

will often be the case in practical applications when higher order terms are negligible. Denote by $N(p)$ the numerator of the integrand in (3.5). This integral becomes

$$M_1^* = \frac{1}{2\pi i} \oint \frac{N(p)\,dp}{C_0 + C_1 p + C_1^* p^{-1}}. \tag{3.11}$$

The roots of the denominator are

$$\begin{aligned} p_1 &= \frac{-C_0 + (C_0^2 - 4C_1C_1^*)^{1/2}}{2C_1} \\ p_2 &= \frac{-C_0 - (C_0^2 - 4C_1C_1^*)^{1/2}}{2C_1}. \end{aligned} \tag{3.12}$$

We shall assume the radical is real, in which case the roots are not on the unit circle. Moreover, if p_1 is inside the unit circle, the other root p_2 is outside.

The integrand of (3.11) is

$$\frac{pN(p)}{C_1p^2 + C_0p + C_1^*} = \frac{H_0p + \sum_{k=1}^{\infty} [H_kp^{k+1} + H_k^*p^{-k+1}]}{C_1(p - p_1)(p - p_2)}. \tag{3.13}$$

There is a residue due to the root p_1 and residues due to the terms $H_k^*p^{-k+1}$ for $k > 1$ corresponding to poles of order $k - 1$ at the origin. The sum of residues inside the unit circle is

$$M_1^* = \frac{p_1N(p_1)}{C_1(p_1 - p_2)} + \sum_{k=2}^{\infty} \frac{H_k^*}{(k-2)!} \left[\frac{d^{k-2}}{dp^{k-2}}\left(\frac{1}{C_1p^2 + C_0p + C_1^*}\right)\right]_{p=0}. \tag{3.14}$$

This expression gives the phase and amplitude of the fundamental Fourier component of the output signal. If the denominator contains more terms than assumed here, we must solve a complex algebraic equation of higher degree and similarly evaluate the residue for these roots inside the circle. If the numerator $N(p)$ contains only one Fourier component

$$N(p) = H_0 + H_1p + H_1^*p^{-1} \tag{3.15}$$

then (3.14) reduces to the very simple form

$$M_1^* = \frac{H_1^* + H_0p_1 + H_1p_1^2}{(C_0^2 - 4C_1C_1^*)^{1/2}}. \tag{3.16}$$

Instead of using the theory of residues, another method of computing the coefficients M_k of the Fourier expansion (3.1) is to write

$$\frac{H_0 + \sum_{k=1}^{\infty} [H_kp^k + H_k^*p^{-k}]}{C_0 + \sum_{k=1}^{\infty} [C_kp^k + C_k^*p^{-k}]} = M_0 + \sum_{k=1}^{\infty} [M_kp^k + M_k^*p^{-k}] \tag{3.17}$$

considering M_k as undetermined, then to multiply both sides of this equation by the denominator and equating coefficients of the same power of p.

ON THE SOLUTION OF A DIFFERENTIAL EQUATION WITH NONLINEARITY APPEARING IN THE SECOND DERIVATIVE OF COMBINED LINEAR AND CUBIC TERMS*

BY

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1. Introduction. One commonly analyzed application of automatic controls to processes is shown schematically in Fig. 1. Here a linear feedback controller is used to control a two-capacity process. The process involves two linear capacities or tanks in this case, and two linear resistances. The analysis of this process is treated in detail in the literature and in the texts on automatic control. The solution is simple and straight-

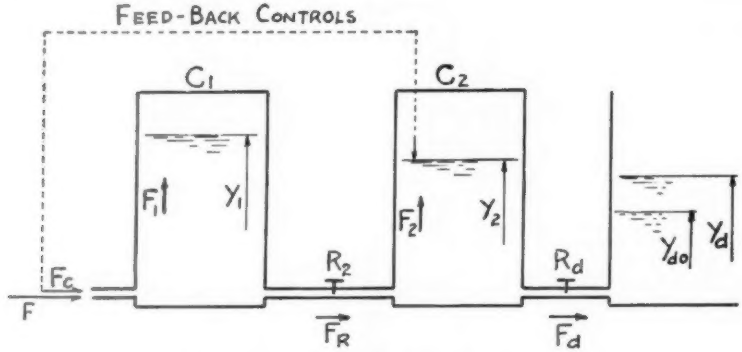

FIG. 1. Linear two-capacity process with controls.

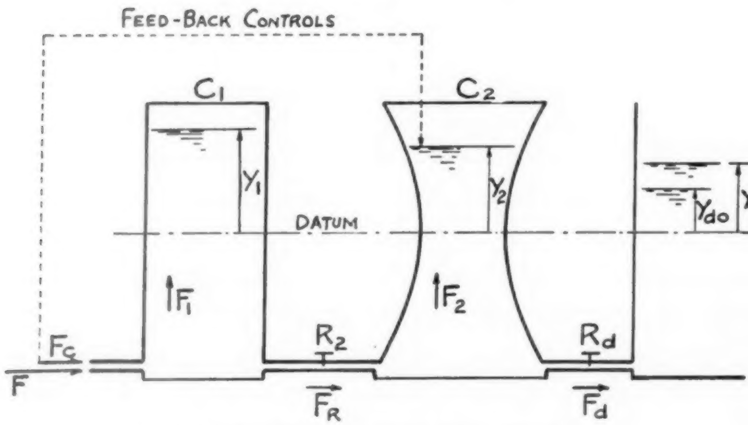

FIG. 2. Single nonlinear capacity of two-capacity process.

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forward. If, however, nonlinear elements are introduced, the problem becomes more complex and cannot be solved in the conventional manner.

If tanks of the form shown in Fig. 2 are used, the capacity is nonlinear. The capacity of a parabolic tank may be expressed by an equation of the form:

$$\text{Volume} = C(x + \alpha x^3 + \gamma x^5). \tag{1}$$

If γ is small, this equation reduces to:

$$\text{Volume} = C(x + \alpha x^3). \tag{2}$$

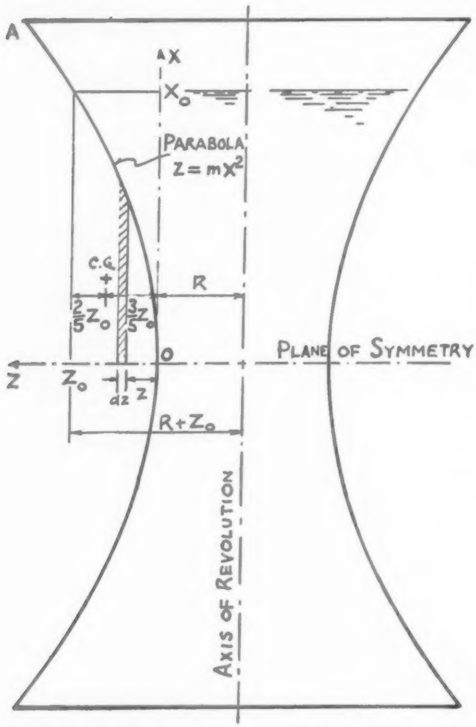

FIG. 3. Nonlinear capacity, neck shaped vessel.

In tanks of the form shown in Fig. 3, α is positive, and for the form shown in Fig. 4, α is negative. This expression also will give the approximate volume of many other tank forms.

If a system consists of tanks, the volume of which may be expressed in the form of Eq. (2), the differential equation which defines the behavior of the system may be written as:

$$C_1 \frac{dy_1}{dt} = a_1 y_1 + a_2 y_2 + a_3 , \tag{3a}$$

$$C_2 \frac{d}{dt}(y_2 + \alpha y_2^3) = b_1 y_1 + b_2 y_2 + b_3 , \tag{3b}$$

where a, b, c with subscripts are all parameters.

The purpose of this paper is to indicate a solution which describes the behavior of such a system. However to facilitate the analysis, it will be convenient to make certain transformations and to discuss the problem in terms of trajectories in the phase-plane.

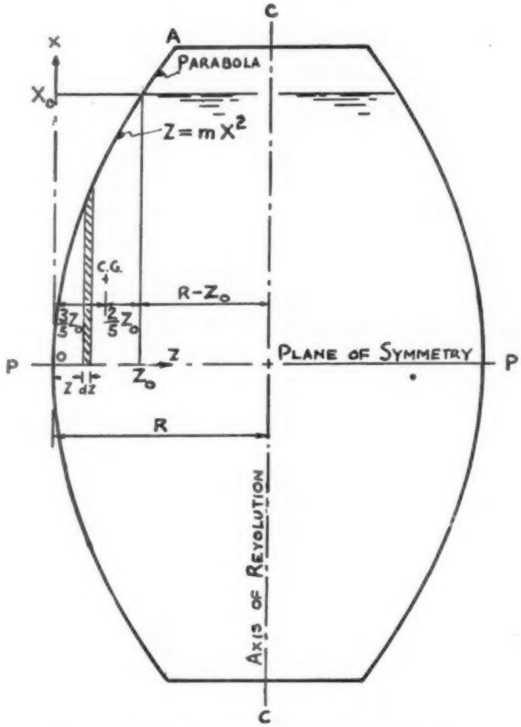

FIG. 4. Nonlinear capacity, barrel shaped vessel.

By solving for y_1 in Eq. (3b), differentiating and substituting the value of y_1 and dy_1/dt into Eq. (3a), it can be shown that:

$$m_1 \frac{d^2}{dt^2}(y_2 + \alpha y_2^3) + m_2 \frac{d}{dt}(y_2 + \alpha y_2^3) + m_3 \frac{d}{dt} y_2 + m_4 y_2 + m_5 = 0, \tag{4}$$

where m's are all parameters.

By a suitable transformation, the differential equation, Eq. (4), may be reduced to the form:

$$\frac{d^2}{d\sigma^2}(Y + \beta Y^3) + M_n \frac{d}{d\sigma}(Y + \beta Y^3) + M_1 \frac{dY}{d\sigma} + Y = r. \tag{5}$$

The quantities β, M_n , M_1 , and r are all parameters. This equation differs from the standard nonlinear differential equations for which solutions are available.

If V is defined as:

$$V = \frac{dY}{d\sigma},$$

then

$$\frac{d}{d\sigma}(Y + \beta Y^3) = (1 + 3\beta Y^2)V,$$

$$\frac{d^2}{d\sigma^2}(Y + \beta Y^3) = V\frac{d}{dY}[(1 + 3\beta Y^2)V].$$

Thus, from Eq. (5)

$$\frac{dV}{dY} = -\left(M_n + \frac{M_1}{1 + 3\beta Y^2}\right) + \frac{r - Y(1 + 6\beta V^2)}{V(1 + 3\beta Y^2)}, \tag{6}$$

where V and Y are the phase-plane variables.

A discussion of the solutions of the key equation (6) in its general form will be temporarily postponed, and a special case where $M_n = M_1 = 0$ will be considered, for in this case a solution in closed form can be obtained.

2. An explicit solution for the special case, $M_n = M_1 = 0$.

When $M_n = M_1 = 0$, note that Eq. (6) takes the form,

$$\frac{V\,dV}{1 + 6\beta V^2} = \frac{r\,dY}{(1 + 3\beta Y^2)(1 + 6\beta V^2)} - \frac{Y\,dY}{1 + 3\beta Y^2}. \tag{7}$$

Equation (7) does not appear to be integrable because the first term on the right-hand side of the equation is a function of both V and Y. However, by introducing a new function ϕ and rearranging the equation, the explicit solution of Eq. (7) may be found.

Define

$$\phi(Y) = 12\beta r\int_{Y_0}^{Y}\frac{dY}{(1 + 3\beta Y^2)(1 + 6\beta V^2)}. \tag{8}$$

Also, from Eq. (7),

$$\phi(Y) = \int_{V_0}^{V}\frac{12V\,dV}{1 + 6\beta V^2} + \int_{Y_0}^{Y}\frac{12\beta Y\,dY}{1 + 3\beta Y^2}, \tag{9}$$

where Y_0 and V_0 are the initial values of Y and V respectively. Equation (9) is readily integrable and

$$\phi(Y) = \ln\frac{1 + 6\beta V^2}{1 + 6\beta V_0^2}\frac{(1 + 3\beta Y^2)^2}{(1 + 3\beta Y_0^2)^2}. \tag{10}$$

Let ϕ_0 be the initial value of ϕ. Using Eq. (10) the value of ϕ_0 may be determined; in fact

$$\phi(Y)_{\mathrm{at}\,\substack{(Y=Y_0)\\(V=V_0)}} = \phi_0 = \ln 1 = 0;$$

thus

$$e^{\phi} = \frac{(1 + 6\beta V^2)(1 + 3\beta Y^2)^2}{Q^2}, \tag{11}$$

where

$$Q^2 = (1 + 6\beta V_0^2)(1 + 3\beta Y_0^2)^2.$$

Combining Eq. (8) and Eq. (11), the variable V may be eliminated and

$$\phi(Y) = 12\beta r \int_{Y_0}^{Y} \frac{\epsilon^{-\phi}(1 + 3\beta Y^2)\, dY}{Q^2}. \tag{12}$$

Differentiating Eq. (12),

$$\frac{d\phi}{dY} = 12\beta r \frac{\epsilon^{-\phi}(1 + 3\beta Y^2)}{Q^2}. \tag{13}$$

Now, the variables are separable and thus by integration

$$\epsilon^{\phi}\Big]_{\phi_0}^{\phi} = \frac{12\beta r}{Q^2}(Y + \beta Y^3)\Big]_{Y_0}^{Y}$$

or

$$\epsilon^{\phi} = 1 + \frac{12\beta r}{Q^2}[(Y + \beta Y^3) - (Y_0 + \beta Y_0^3)]. \tag{14}$$

Combining Eqs. (11) and (14), the solution of Eq. (7) may be obtained:

$$(1 + 6\beta V^2)(1 + 3\beta Y^2)^2 = Q^2 + 12\beta r[(Y + \beta Y^3) - (Y_0 + \beta Y_0^3)]$$

or

$$V^2 = \frac{S^2 - (Y^2 + 1.5\beta Y^4) + 2r(Y + \beta Y^3)}{(1 + 3\beta Y^2)^2}, \tag{15}$$

where

$$S^2 = Y_0^2 + 1.5\beta Y_0^4 + V_0^2(1 + 3\beta Y_0^2)^2 - 2r(Y_0 + \beta Y_0^3) \tag{16}$$

is a function of the starting conditions. Equation (15) together with Eq. (16) yields the solution of the differential Eq. (7) in closed form.

3. Bounded and unbounded solutions for $r = 0$ when $M_n = M_1 = 0$. Bounded and unbounded solutions in the special case under consideration will be investigated. For any real value of Y and any positive value of β it may be seen that $(1 + 3\beta Y^2)^2 > 1$ and from Eq. (11), setting $r = 0$, that

$$V^2 < S^2 - (Y^2 + 1.5\beta Y^4),$$

or

$$V^2 + Y^2 + 1.5\beta Y^4 < S^2, \tag{17}$$

where

$$S^2 = Y_0^2 + 1.5\beta Y_0^4 + V_0^2(1 + 3\beta Y_0^2)^2.$$

Thus V and Y are bounded in the phase plane as shown in Eq. (17) where the left-hand quantity has to be smaller than a positive real number S^2.

However, for a negative value of β the situation may be quite different. A plot of the solution of Eq. (7) on the phase-plane from Eq. (15) and Eq. (16) will demonstrate

this. It is clear that the solution of Eq. (7) depends on the starting conditions in the phase-plane. A numerical example where β is negative will illustrate this feature.

Assume

$$\beta = -\frac{4}{300}, \qquad r = 0,$$

then

$$V^2 = \frac{S^2 - Y^2 + .02Y^4}{(1 - .04Y^2)^2}.$$

These phase plane trajectories are plotted in Fig. 5. At $S^2 = 12.5$, the value of $| V |$ is 3.535 for every value of Y, except $| Y | = 5.00$. At $| Y | = 5.00$, V is indeterminate. The system is bounded provided the initial point (Y_0, V_0) falls within the rectangular box $| Y | \leq 5$, $| V | \leq 3.535$. An initial point falling outside the box gives rise to an unbounded trajectory.

For example, if the starting conditions are given as

$$Y_0 = +4, \qquad V_0 = +4$$

we can see that the system is not bounded.

4. The effect of changing the parameter β. The choice of any particular value for β, for example $\beta = \pm 4/300$, in no way limits the generality of the treatment, for the characteristic of the solution for any other value of β can be obtained by introducing a scale factor, i.e., a transformation parameter may be used to make the coefficient β equal to $\pm 4/300$. More precisely, if this parameter is denoted by L, let $Y = LZ$, then Eq. (5) may be written as

$$\frac{d^2}{d\sigma^2}(Z + \beta L^2 Z^3) + M_n \frac{d}{d\sigma}(Z + \beta L^2 Z^3) + M_1 \frac{dZ}{d\sigma} + Z = \frac{r}{L},$$

or

$$\frac{d^2}{d\sigma^2}(Z + \xi Z^3) + M_n \frac{d}{d\sigma}(Z + \xi Z^3) + M_1 \frac{dZ}{d\sigma} + Z = R, \tag{18}$$

where

$$\xi = \beta L^2 \quad \text{and} \quad R = \frac{r}{L}. \tag{19}$$

Hence ξ may be made equal to either $+ 4/300$ or $-4/300$ by a suitable scale factor L^2 and R can be calculated correspondingly. The solution of Eq. (5) will be $Y = LZ$, where Z is readily obtainable by comparing Eq. (18) with Eq. (5). These remarks apply not only to the general equation, but also to the special equation arising when $M_n = M_1 = 0$.

5. Bounded and unbounded solutions for $r \neq 0$ when $M_n = M_1 = 0$. In Sec. 3 it was shown that a solution would be bounded when $r = 0$ if the initial values (Y_0, V_0) fell within a certain rectangle (see Fig. 5). When $r \neq 0$ this bounded region changes shape. Indeed the bounded region for Eq. (7) with a negative nonlinearity has a horseshoe shape.

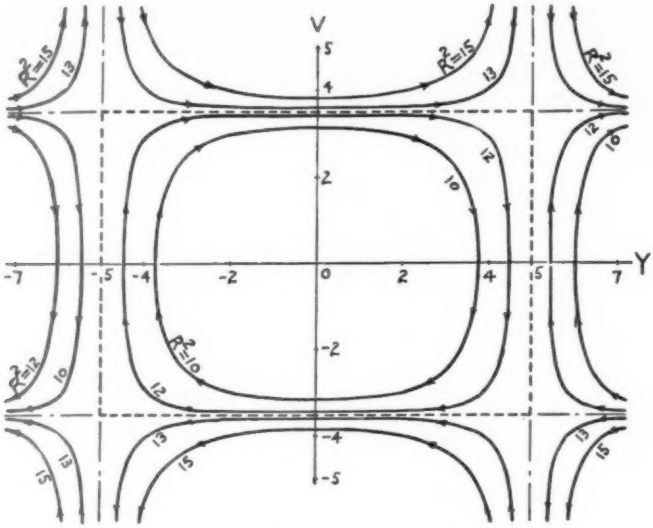

FIG. 5. Negative nonlinearity for $\beta = -4/300$ and $r = 0$.

To demonstrate this, Eq. (15) may be re-examined, keeping in mind that a bounded value of V is required as $1 + 3\beta Y^2$ approaches zero. In this case the numerator of the righthand side of Eq. (15) also approaches zero

$$S^2 - (Y^2 + 1.5\beta Y^4) + 2r(Y + \beta Y^3) \to 0. \tag{20}$$

This is the criterion for a boundary of the bounded region.

By substituting $Y^2 = -1/3\beta - \epsilon$ into Eq. (20), we have

$$S^2 = -\frac{1}{6\beta} - \frac{4r}{3}\left(-\frac{1}{3\beta}\right)^{1/2} + \epsilon_1 , \tag{21}$$

where ϵ and ϵ_1 are all infinitesimals.

Since β is negative, β' may be defined as $\beta' \equiv -\beta$. Using this value, β', and the value of S^2 from Eq. (16), Eq. (21) becomes

$$V_0^2 = \frac{(1/6\beta') - (4r/3)(1/3\beta')^{1/2} + 2r(Y_0 - \beta' Y_0^3) - Y_0^2 + 1.5\beta' Y_0^4}{(1 - 3\beta' Y_0^2)^2}. \tag{22}$$

The above equation determines the bounded region for Eq. (7).

It is interesting to note that if $Y_0^2 = 1/3\beta'$, V_0^2 is indeterminate. However, the value of V_0^2 can be found as $Y_0^2 \to 1/3\beta'$ by differentiating the numerator and denominator of Eq. (22) with respect to Y_0 ; in fact

$$\lim_{Y_0 \to (1/3\beta')^{1/2}} V_0^2 = \frac{2r - 6\beta' r Y_0^2 - 2Y_0 + 6\beta' Y_0^3}{2(1 - 3\beta' Y_0^2)(-6\beta' Y_0)} = \frac{Y_0 - r}{6\beta' Y_0}. \tag{23}$$

If the solutions to Eqs. (22) and (23) are plotted for $r = 0.1, 1.0, 2.0, 4.0$ at $\beta = -\beta' = -4/300$, the boundary is seen to be of U or horseshoe shape as shown in Fig. 6.

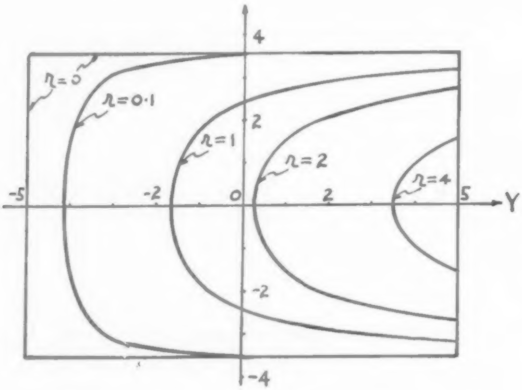

FIG. 6. Bounded regions for different r, $\beta = -4/300$.

At a particular value of r, the solution is bounded when any set of starting conditions lay within the horseshoe region; otherwise it is unbounded. The phase plane diagram for Eq. (15) is also given for $r = 1.0$; see Figs. 7 and 8.

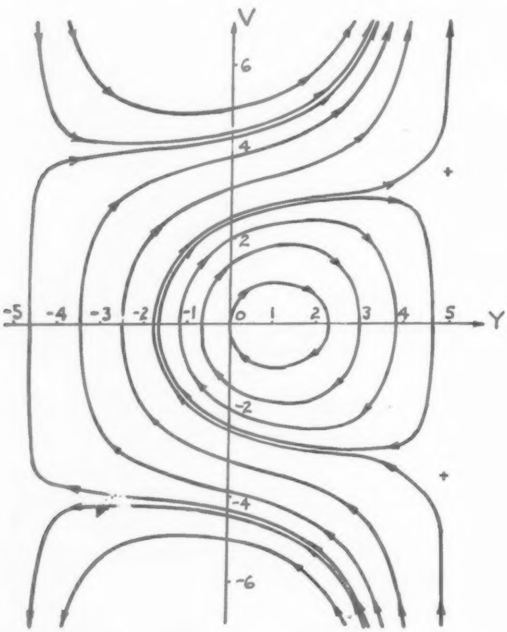

FIG. 7. Negative nonlinearity for $\beta = -4/300$ and $r = 1.0$.

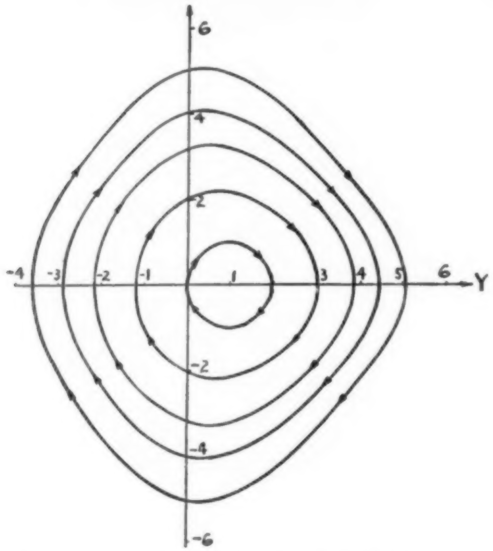

FIG. 8. Positive nonlinearity for $\beta = +4/300$ and $r = 1.0$.

6. The period and frequency when $M_n = M_1 = 0$. The period of a bounded solution of Eq. (7) is in general an elliptic integral. This can be seen by rearranging Eq. (15) in the form

$$\int_0^\sigma d\sigma = \int_0^Y \frac{(1 + 3\beta Y^2)\, dY}{[S^2 - (Y^2 + 1.5\beta Y^4) + 2r(Y + \beta Y^3)]^{1/2}}. \tag{24}$$

If numerical values are assigned to the parameters, the integral of Eq. (24) may be evaluated by referring to any standard text on elliptic integrals.

If $r = 0$, the general solution can be found by considering

$$\frac{d^2}{d\sigma^2}(Y + \beta Y^3) + Y = 0. \tag{25}$$

Expressing σ in terms of Y, Eq. (25) may be written in the form

$$\int_0^p d\sigma = \int_0^{Y_m} \frac{(1 + 3\beta Y^2)\, dY}{[S^2 - Y^2 - 1.5\beta Y^4]^{1/2}}, \tag{26}$$

where

$$S^2 = Y_0^2 + 1.5\beta Y_0^4 + V_0^2(1 + 3\beta Y_0^2)^2.$$

The solution of Eq. (26) is a combined elliptic integral of the first and second kind. If β is negative, then the quarter period will be

$$p = \left(\frac{Y_m}{S} - \frac{2S}{Y_m}\right)K(k) + \frac{2S}{Y_m}E(k), \tag{27}$$

where

$$Y_m^2 = \frac{1 - [1 - 6\beta' S^2]^{1/2}}{3\beta'}, \qquad k = \left[\frac{Y_m^2 - S^2}{S^2}\right]^{1/2}, \tag{28}$$

Y_m being the maximum value of Y at $V = 0$, if bounded and $\beta' \equiv -\beta$. K denotes the complete elliptical integral of the first kind. E denotes the complete elliptical integral of the second kind. The general solution of Eq. (26) will be in the form

$$\sigma = \left(\frac{Y_m}{S} - \frac{2S}{Y_m}\right)F(\theta, k) + \frac{2S}{Y_m}E(\theta, k), \tag{29}$$

where

$$\theta = \sin^{-1}\frac{Y}{Y_m}. \tag{30}$$

If β is positive, the quarter period will be

$$p = +\left[2\left(\frac{S}{Y_m}\right)^2 - 1\right]^{1/2}[2E(k) - K(k)], \tag{31}$$

where

$$Y_m^2 = \frac{-1 + [1 + 6\beta S^2]^{1/2}}{3\beta}, \qquad k = \left[\frac{S^2 - Y_m^2}{2S^2 - Y_m^2}\right]^{1/2}, \tag{32}$$

Y_m being the maximum value of Y at $V = 0$. The general solution of Eq. (26) will be in the form

$$\sigma = \left[2\left(\frac{S}{Y_m}\right)^2 - 1\right]^{1/2}\left[2E\left(\frac{\pi}{2}, k\right) - 2E(\theta, k) - K\left(\frac{\pi}{2}, k\right) + F(\theta, k)\right], \tag{33}$$

where

$$\theta = \cos^{-1}\frac{Y}{Y_m}. \tag{34}$$

The frequency may be calculated for both negative and positive β as

$$\omega = \frac{\pi/2}{p}.$$

The frequencies for various values of Y_m and for $\beta = \pm 4/300$ are plotted in Fig. 9. It is interesting to compare Eq. (25) with the Duffing's equation,

$$\frac{d^2}{d\tau^2}X + X + \lambda X^3 = 0, \tag{35}$$

where the period is an elliptic integral of the first kind only. If β and λ are small, the approximation $\lambda = -\beta = \beta'$ may be used.

Therefore the negative nonlinearity of Eq. (25) corresponds to the positive nonlinearity of the Duffing's equation, and vice versa. This can be seen as the positive branch curves towards the lower frequencies just as the negative branch does for the Duffing's equation. However, the frequency will be entirely different as starting amplitude gets higher.

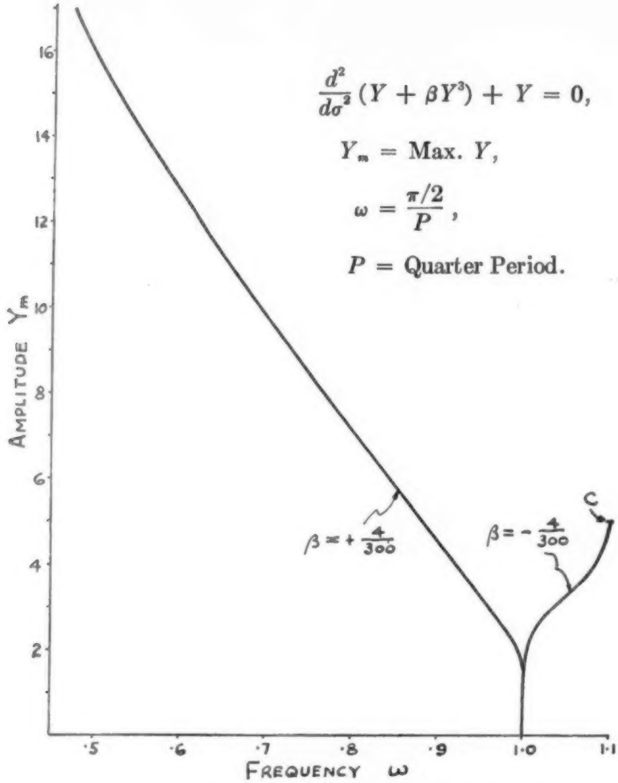

FIG. 9. Frequency amplitude response. c denotes cut-off point at $Y_m = 5.0$.

For a very high amplitude the frequency slows down for a positive β, while there is no bounded solution for a negative λ with a very large amplitude in the Duffing's equation. For a negative β, the amplitude is restricted and not more than $(1/3\beta')^{1/2}$, while there is always a frequency for any positive λ at high amplitude for Eq. (35).

7. Solution with damping terms. With all these details in mind for the characteristic of solutions of Eq. (7) in the special case $M_n = M_1 = 0$, let us turn now to the characteristic of solutions of the general equation (6). Note first of all that the slopes of trajectories in the phase plane, as specified by Eq. (7), must be decreased by an amount $M_n + [M_1/(1 + 3\beta Y^2)]$ if the slope of the trajectories is to be that specified by Eq. (6). In other words, when M_n and M_1 are not zero, damping occurs and the slopes in the phase plane must be adjusted accordingly. Conceivably, then, one would begin with a solution of Eq. (7), sketch in a number of isoclines (i.e., a number of tangent line elements); adjust the slopes and draw new isoclines for Eq. (6), and then sketch in the trajectories for Eq. (6) and have a picture of the performance in the large for the general case.

For the general case, however, it was more convenient to use an analog computer which sketched the trajectories for the following cases:

$$\beta = -\frac{4}{300}, \qquad r = 2.0$$

$$M_1 = 0.2, M_n = 0 \quad \text{in Fig. 10}$$

$$M_1 = 1.0, M_n = 0 \quad \text{in Fig. 11}$$

$$M_1 = 0, \quad M_n = 0.2 \text{ in Fig. 12}$$

$$M_1 = 0, \quad M_n = 1.0 \text{ in Fig. 13}$$

Note that if $\beta > 0$, the quantity $M_n + [M_1/(1 + 3\beta Y^2)]$ representing the change in slope is always positive and the effect will be the creation of inward spirals toward the value r. If $\beta < 0$, the change in slope is also positive for bounded solutions since the quantity $(1 + 3\beta Y^2)$ is always positive for bounded solutions without damping. Moreover, $(1 + 3\beta Y^2) < 0$ is not possible in a physical situation.

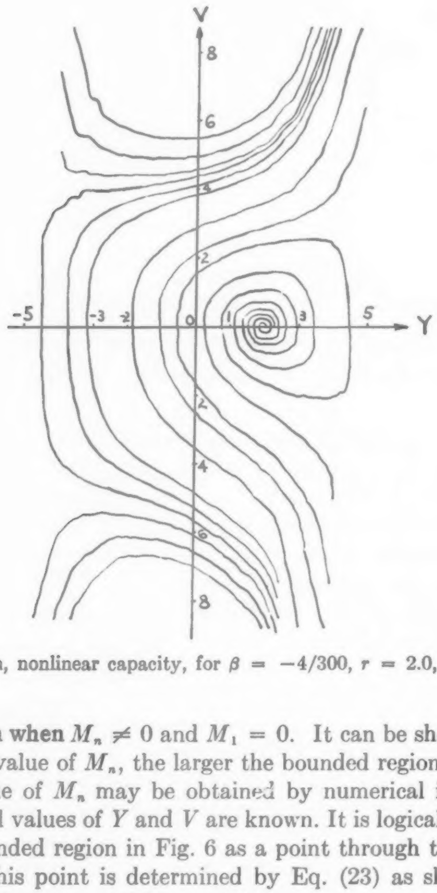

FIG. 10. Computer solution, nonlinear capacity, for $\beta = -4/300$, $r = 2.0$, $M_1 = 0.2$, and $M_n = 0$.

8. Bounded solution when $M_n \neq 0$ and $M_1 = 0$. It can be shown from Figs. 12 and 13 that the higher the value of M_n, the larger the bounded region. This bounded region for any particular value of M_n may be obtained by numerical integration of Eq. (6) provided that the initial values of Y and V are known. It is logical to consider the upper right corner of the bounded region in Fig. 6 as a point through the integral curve. For any given value of r this point is determined by Eq. (23) as shown in the following example, where $r = 2$, $Y_c = 5.0$, and $V_c = 2.7387$.

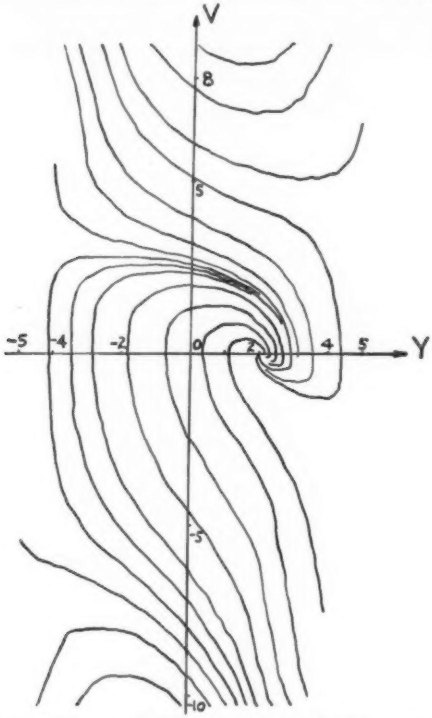

FIG. 11. Computer solution, nonlinear capacity, for $\beta = -4/300$, $r = 2.0$, $M_1 = 1.0$ and $M_s = 0$.

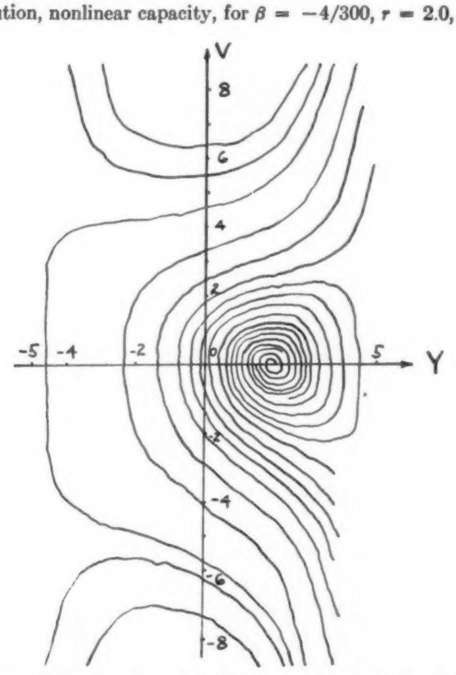

FIG. 12. Computer solution, nonlinear capacity, for $\beta = -4/300$, $r = 2.0$, $M_1 = 0$ and $M_s = 0.2$.

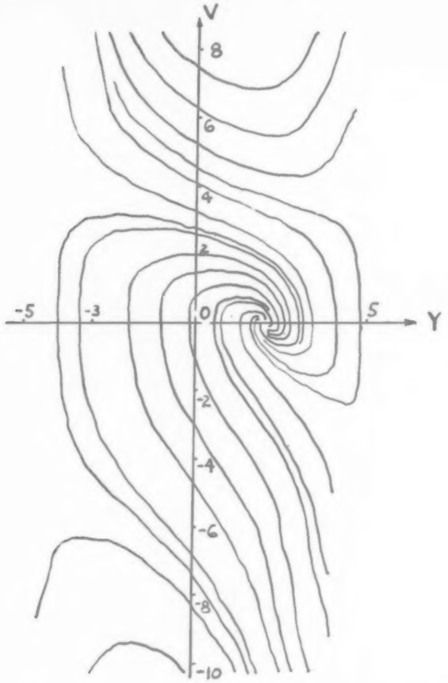

FIG. 13. Computer solution, nonlinear capacity, for $\beta = -4/300$, $r = 2.0$, $M_1 = 0$ and $M_n = 1.0$.

A closer examination of Fig. 6 and Eq. (23) indicates that this corner point obviously has two distinct slopes—one being infinite, the other being finite, depending upon the direction of approach to the point. To determine the finite slope Eq. (6) is rewritten in the following form:

$$\lim_{1+3\beta Y^2\to 0} \frac{dV}{dY} = -M_n + \lim_{1+3\beta Y^2\to 0} \frac{r - Y(1 + 6\beta V^2)}{V(1 + 3\beta Y^2)}.$$

By denoting the limit values of V and Y as V_c and Y_c respectively, it can be shown that

$$\left(\frac{dV}{dY}\right)_c = -(1 + 6\beta V_c^2)(18\beta V_c Y_c)^{-1} - M_n/3. \tag{36}$$

The limiting value of the second derivative of V with respect to Y can also be expressed as

$$-\left(\frac{d^2V}{dY^2}\right)_c = V_c^{-1}\left(\frac{dV}{dY}\right)_c^2 + (1.25Y_c^{-1} + 0.50M_nV^{-1})\left(\frac{dV}{dY}\right)_c + 0.25M_nY_c^{-1}. \tag{37}$$

For the same example with $r = 2$ and $M_n = 1.0$, one obtains

$$\left(\frac{dV}{dY}\right)_c = -.2116, \qquad \left(\frac{d^2V}{dY^2}\right)_c = .0251.$$

To start the integral curve the Taylor's expansion with three terms is adequate.

$$V(Y_e + h) = V_e + h\left(\frac{dV}{dY}\right)_e + \frac{h^2}{2}\left(\frac{d^2V}{dY^2}\right)_e + \cdots$$

where h is the increment of Y. The mth approximation of f may be defined as

$$\frac{dV}{dY} = f_m(Y) = -M_n + \frac{r - Y_m(1 + 6\beta V_m^2)}{V_m(1 + 3\beta Y_m^2)}. \tag{38}$$

The above formulae together with the following Simpson's rule constitutes the method of successive approximation in numerical integration.

$$V_m(a + 2h) = V(a) + \frac{h}{3}\,[f(a) + 4f(a + h) + f_{m-1}(a + 2h)].$$

TABLE 1

Bounded region with damping term M_n for $r = 2$, $M_n = 1.0$, $M_1 = 0$

Y	V	dV/dY	Y	V	dV/dY
5.0	2.7387	−.2116	−5.0	−4.1833	−.4130
4.8	2.7815	−.2166	−4.8	−4.2671	−.4254
4.6	2.8253	−.2216	−4.6	−4.3535	−.4377
4.4	2.8702	−.2283	−4.4	−4.4425	−.4542
4.2	2.9165	−.2336	−4.2	−4.5350	−.4680
4.0	2.9637	−.2401	−4.0	−4.630	−.484
3.6	3.063	−.253	−3.6	−4.830	−.519
3.2	3.167	−.269	−3.2	−5.045	−.558
2.8	3.278	−.286	−2.8	−5.277	−.603
2.4	3.396	−.306	−2.4	−5.528	−.655
1.6	3.662	−.356	−1.6	−6.102	−.787
0.8	3.972	−.429	−0.8	−6.802	−.976
0	4.359	−.541	0	−7.689	−1.260
−0.8	4.857	−.727	0.8	−8.865	−1.721
−1.6	5.565	−1.073	1.6	−10.540	−2.545
−2.4	6.677	−1.809			

The same technique can be applied to the lower left corner of the bounded region in Fig. 13. The results are listed in Table 1 and plotted in Fig. 14. This is to say that two integral curves determine this bounded region.

On the other hand, for $M_n = 0.2$ and $M_1 = 0$ there exists only one integral curve. It is sufficient to determine the bounded region by this simple curve alone.

9. The general bounded solutions. The value of $M_1\ (1 + 3\beta Y^2)^{-1}$ approaches infinity as $(1 + 3\beta Y^2)$ becomes zero. However, the corner point with two distinct slopes shifts to a new position by letting

$$-Y_e(1 + 6\beta V_e^2) - M_1V_e + r = 0,$$

from which is obtained

$$V_e = -\frac{M_1}{12\beta Y_e} \pm \left[\left(\frac{M_1}{12\beta Y_e}\right)^2 + \frac{r - Y_e}{6\beta Y_e}\right]^{1/2}. \tag{39}$$

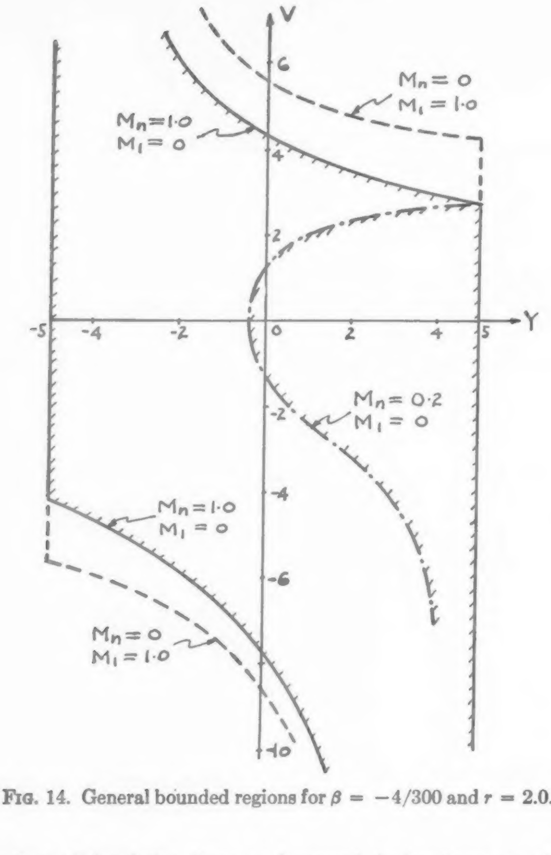

FIG. 14. General bounded regions for $\beta = -4/300$ and $r = 2.0$.

The limiting value of the finite slope and second derivatives are also obtained as

$$\left(\frac{dV}{dY}\right)_c = -(1 + 6\beta V_c^2 + 6\beta V_c Y_c M_n)(18\beta V_c Y_c + M_1)^{-1}, \tag{40}$$

$$-\left(\frac{d^2 V}{dY^2}\right)_c = \left[V_c^{-1}\left(\frac{dV}{dY}\right)_c^2 + (1.25Y_c^{-1} + 0.50M_n V^{-1})\left(\frac{dV}{dY}\right)_c + 0.25M_n Y_c^{-1}\right] \cdot [1 + M_1/24\beta Y_c V_c]^{-1}. \tag{41}$$

The same method is used in numerical integration. The results are plotted in Fig. 14 for $r = 2$, $M_1 = 1.0$, and $M_n = 0$. Figure 14 indicates that the bounded region with $M_1 = 1.0$ and $M_n = 0$ is bigger than that with $M_n = 1.0$ and $M_1 = 0$. Thus it can be concluded that the M_1 term yields more weight in damping than the M_n term. For the same type of damping, the lower the value of the M_n or the M_1 term, the smaller the bounded region. This can be shown by comparing Figs. 10 and 11, or Figs. 12 and 13.

For $M_1 = 0$ and M_n being finite, there are always some unbounded solutions of the differential equation. This is also true for $M_n = 0$ and M_1 being finite, except the value of V_c being higher.

10. Further physical applications. An electric circuit with linear resistance and capacitance in series with a coil having inserted magnetic material can serve as a good example for illustrating more explicitly the results in physical terms. The coil has N

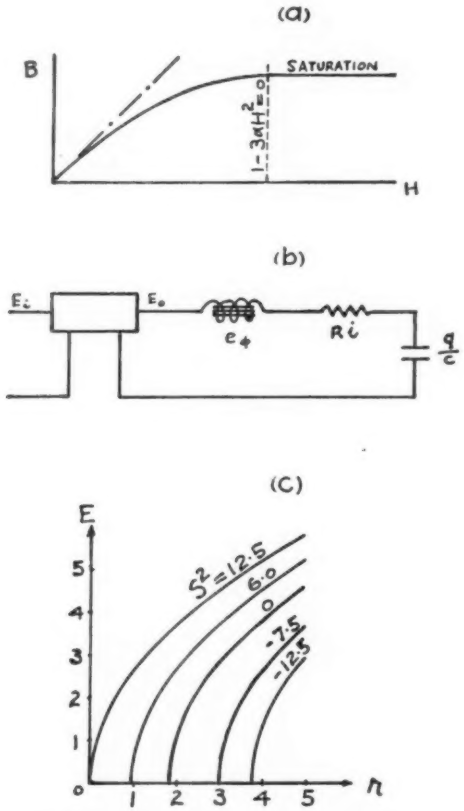

FIG. 15. Nonlinear circuit: a) saturation, b) circuit elements, c) E denotes $V(1 + 3\beta Y^2)$ as $1 + 3\beta Y^2 \to 0$

turns with a magnetic flux ϕ, thus the emf across the terminals of the coil as shown in Fig. 15(a) is

$$e_\phi = \frac{d}{dt}(N\phi).$$

The differential equation can be written as

$$\frac{d(N\phi)}{dt} + R\frac{dq}{dt} + \frac{q}{c} = E_0 . \tag{42}$$

To solve the above equation it is necessary to determine the relation between ϕ and q. If the flux density is B and the magnetising force is H, the relation between B

and H before saturation may be approximated by

$$B = K(H - \alpha H^3), \tag{43}$$

where

$$K = \text{constant}, \ \alpha = \text{nonlinear parameter}.$$

Let μ be the incremental permeability of the material, then by differentiating Eq. (43)

$$\mu = \frac{dB}{dH} = K(1 - 3\alpha H^2).$$

If H is small, μ is approximately equal to K as its limiting value. The flux density is saturated at

$$\frac{dB}{dH} = \mu(1 - 3\alpha H^2) = 0.$$

Thus the $B - H$ relation can be defined only within the region

$$1 - 3\alpha H^2 > 0,$$

or

$$H^2 < 1/3\alpha. \tag{44}$$

Substituting the value of $H = Ni1^{-1}$ and $B = \phi A^{-1}$.

$$\phi = S^{-1}N(i - \gamma i^3), \tag{45}$$

where A is the cross section area of the material, 1 the length of the magnetic circuit, i the current, $S = 1K^{-1}A^{-1}$ the incremental reluctance, and $\gamma = \alpha N^2 1^{-2}$ a nonlinear parameter.
Thus,

$$e_\phi = \frac{d}{dt}(N\phi) = L\frac{d}{dt}(i - \gamma i^3), \tag{46}$$

where $L = S^{-1}N^2$ = inductance for the linear portion.
By substituting e_ϕ in Eq. (42), the differential equation becomes

$$L\frac{d}{dt}(i - \gamma i^3) + Ri + \frac{q}{c} = E_0 . \tag{47}$$

The differential of the above equation will yield,

$$L\frac{d^2}{dt^2}(i - \gamma i^3) + R\frac{di}{dt} + \frac{i}{c} = \frac{dE_0}{dt}. \tag{48}$$

For a step change of E_0 $dE_0/dt = 0$ it can be seen that this is reduced to

$$\frac{d^2}{d\sigma^2}(i - \gamma i^3) + (RL^{-1/2}C^{1/2})\frac{di}{d\sigma} + i = 0, \tag{49}$$

where

$$\sigma = (LC)^{-1/2}t,$$

or

$$\frac{d^2}{d\sigma^2}(Y + \beta Y^3) + M_1 \frac{dY}{d\sigma} + Y = 0, \tag{50}$$

where $i = IY$, $\beta = -\gamma I^2 = -4/300$, $M_1 = RL^{-1/2}C^{1/2}$, and I is a reference current.

For a control device the system sometimes follows a ramp type signal defined by

$$E_0 = \tau t, \tag{51}$$

where τ is a constant.

The system will have a differential equation

$$\frac{d^2}{d\sigma^2}(i - \gamma i^3) + (RL^{-1/2}C^{1/2})\frac{di}{dt} + i = \tau \tag{52}$$

with the starting conditions at $t = 0$, $i = i_0$, $di/dt = 0$. Equation (52) can be expressed in nondimensional form as

$$\frac{d^2}{d\sigma^2}(Y + \beta Y^3) + M_1 \frac{dY}{d\sigma} + Y = r, \tag{53}$$

where $r = \tau I^{-1}$ at $\sigma = 0$, $Y = Y_0 = i_0 I^{-1}$, $dY/d\sigma = 0$.

By knowing the values of L, R, C, γ, τ and i_0 it is possible to compute the parameters L, M_1, r and Y_0 with the fixed value of $\beta = -4/300$. To determine whether the solution is bounded it is a simple matter to examine the starting conditions in the phase plane curves for the above nondimensional parameters.

The definition of boundedness as given in previous discussions is entirely mathematical. The unbounded solution was interpreted as the value of V not bounded. However, close examination of Eq. (15) indicates that the product of V and $(1 + 3\beta Y^2)$ is bounded. This can be written as

$$V(1 + 3\beta Y^2) = [S^2 - (Y^2 + 1.5\beta Y^4) + 2r(Y + \beta Y^3)]^{1/2}. \tag{54}$$

The curves in Fig. (15) give the values of $V(1 + 3\beta Y^2)$ as $(1 + 3\beta Y^1)$ approaches zero. It can be proved that the damping terms M_1 and M_n do not affect these values. Equation (6) may be rearranged as

$$\frac{d}{d\sigma}(Y + \beta Y^3) = V(1 + 3\beta Y^2) = \frac{r - Y(1 + 6\beta V^2)}{(dV/dY) + M_n + M_1(1 + 3\beta Y^2)^{-1}}. \tag{55}$$

As $(1 + 3\beta Y^2)$ approaches zero, V^2 and dV/dY are of higher order than $(1 + 3\beta Y^2)^{-1}$.

The importance of introducing the above analysis is that the voltage e_ϕ is always finite at the saturation flux. This is because e_ϕ can be expressed in the form of Eq. (55). Physically the rate of change of Y, (i.e., V), rises very rapidly as Y approaches $(1-1/3\beta)^{1/2}$, but the value of Y is only defined within this region.

To carry the physical problem to a new region, the final condition of the nonlinear differential equation is of interest.

$$\frac{q}{c} = \alpha t_f - Ri_f - e_f\,, \tag{56}$$

where i_f and e_f are final values of i and e_ϕ respectively, and t_f is the time duration for the current i becoming i_f. The new region has a horizontal line on the B *vs.* H curve after the flux is saturated, ϕ being constant. It is therefore concluded that the value of e_ϕ suddenly drops to zero outside the nonlinear region. The system will then become a linear RC circuit with the same ramp input

$$R\frac{dq}{dt} + \frac{q}{c} = rt. \tag{57}$$

The starting conditions of this linear differential equation are the same as the final conditions of the nonlinear differential equation.

11. Conclusions. The solution of the system of differential equations (3a) and (3b) can be conveniently represented in the YV-phase plane and various sets of graphs made to show the effect of varying the parameters β, r, M_n and M_1. For a positive β the system is always bounded and the frequency tends to be slower than a corresponding linear case. For a negative β with no damping the system is bounded within a horseshoe region and the frequency tends to be faster. If the system is bounded without damping, it is always bounded with damping. The effect will be a spiraling towards the value r in the phase plane.

In a physical system an unbounded solution may be re-defined by extending to a new region beyond its boundary. The analysis can be applied to liquid level controlled systems or circuits with flux saturation.

EXPANSIONS OF THE IRREGULAR COULOMB FUNCTION*

BY

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Abstract. Convergent and asymptotic series in the energy are derived for an irregular Coulomb function appropriate to both attractive and repulsive potentials. The coefficients in these series are Bessel functions. For repulsive potentials the derivation yields series in agreement with those given by Breit and others. Some of the series for attractive potentials have been given by Kuhn. The present general derivation and some of the series are, however, new. Tables of the coefficients in the series for the attractive potentials for values of the parameters of interest in solid state energy band calculations have been prepared and are available.

I. Introduction. Breit and Hull[1] and Abramowitz[2] have shown that the irregular Coulomb function G_L can be expanded in an asymptotic power series in the energy which is valid for repulsive Coulomb fields. Jackson and Blatt[3] have given a convergent series of similar form for G_0, and their derivation can evidently be extended to higher L. In the present note we shall derive convergent and asymptotic series for an irregular function appropriate to both attractive and repulsive potentials and shall obtain the results of the above authors as a special case. A review of the numerical treatment of Coulomb functions has been given recently by Fröberg (Rev. Mod. Phys. **27**, 399 (1955)).

The investigation leading to these general results was undertaken in order to establish formulas for the convenient numerical calculation of an irregular function for attractive Coulomb fields and positive and negative energies of small absolute value. Such calculations are an essential preliminary to the theoretical investigation with the "Quantum Defect Method"[4] of the energy band structure of the alkali metals, a program of research, which is being continued by Brooks and the author. Therefore, we shall use a notation in this paper which is most appropriate to work with an attractive Coulomb field, but in an appendix we shall show the connections with the notation used by Breit and others in their work with a repulsive field and shall show that our results agree with theirs.

Kuhn[5] has already given the asymptotic series that will be derived in this paper, but his presentation is marred by the fact that he did not realize that these irregular series do not converge. Our convergent series are, however, new. We shall assume that the reader is familiar with Kuhn's paper.

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[2]M. Abramowitz, J. Math. and Phys. **33**, 111 (1954)

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[4]H. Brooks and F. S. Ham, Phys. Rev. (to be published)

[5]T. S. Kuhn, Quart. Appl. Math. **9**, No. 1 (1951)

A set of tables of the coefficients in the series for the regular and irregular functions for the attractive Coulomb field has been prepared for values of the parameters of interest in the solid state calculations. These extend and correct the tables given by Kuhn[5] and are accompanied by a more detailed discussion of the formulas than can be given here. They contain also tables of various quantities which appear in the formulas and which will not be written out explicitly here. These tables are now available and may be obtained from the author or from the Division of Applied Science, Harvard University[6].

II. Convergent series representations of Coulomb functions for non-integral values of $(2L + 1)$. The hydrogenic radial wave equation in Rydberg units for an attractive potential takes the form[5]

$$\frac{d^2U}{dr^2} + \left[-\frac{1}{n^2} + \frac{2}{r} - \frac{L(L+1)}{r^2}\right]U = 0, \tag{1}$$

where r is the radial coordinate, U the radial part of the wave function multiplied by r, L the angular momentum quantum number, and $E = -1/n^2$ the energy. Yost, Wheeler and Breit[7] and Kuhn[5] have shown that solutions of (1) exist which have the form

$$U^{(L,n)}(z) = \sum_{k=0}^{\infty} n^{-2k} U_k^{(L)}(z), \tag{2}$$

where $z = (8r)^{1/2}$. If we set

$$U_k^{(L)}(z) = (z/2) V_k^{(L)}(z), \tag{3}$$

we find that the $V_k^{(L)}(z)$ satisfy the simultaneous differential equations

$$\nabla_L V_0^{(L)}(z) = 0, \tag{4a}$$

$$\nabla_L V_k^{(L)}(z) = (z/2)^4 V_{k-1}^{(L)}(z), \tag{4b}$$

where

$$\nabla_L = z^2\left(\frac{d^2}{dz^2}\right) + z\left(\frac{d}{dz}\right) + z^2 - (2L+1)^2. \tag{5}$$

Kuhn has shown from the recurrence relations satisfied by the Bessel functions that if $C_m(z)$ is a linear combination of the Bessel function $J_m(z)$ and the Weber function $Y_m(z)$

$$C_m(z) = AJ_m(z) + BY_m(z), \tag{6}$$

A and B being independent of m, then

$$\nabla_L\left\{\frac{2L+2+q}{4(2+q)}(z/2)^{q+2}C_{2L+3+q}(z) - \frac{1}{4(3+q)}(z/2)^{q+3}C_{2L+4+q}(z)\right\} = (z/2)^{q+4}C_{2L+1+q}(z), \tag{7}$$

and solutions of (4) may be generated if we choose

$$V_0^{(L)}(z) = C_{2L+1}(z). \tag{8}$$

[6] F. S. Ham, *Tables for the calculation of Coulomb wave functions*, Technical Report No. 204, Cruft Laboratory, Harvard University

[7] F. L. Yost, J. A. Wheeler and G. Breit, Phys. Rev. 49, 174 (1936)

In particular, if in (6) we choose $A = 1$, $B = 0$, and if we then determine $V_k^{(L)}(z)$ from (7) for $k \geq 1$, so that no $V_k^{(L)}(z)$ for $k \geq 1$ when expanded in powers of $(z/2)$ contains the power $(z/2)^{2L+1}$, we obtain the series

$$J_{2L+1}^n(z) = (2/z)\ {}^0U_*^{(L,n)}(z) = \sum_{k=0}^{\infty} n^{-2k} \sum_{q=2k}^{3k} a_{k,q}^{(L)} (z/2)^q J_{2L+1+q}(z). \tag{9}$$

It is shown in Appendix I that this series converges absolutely and uniformly for $|z| \leq R$ and $|n| \geq n_0$, where R and n_0 are arbitrary positive numbers.

Similarly, we may show that

$$\nabla_L\left\{\frac{-2L+q}{4(2+q)}\left(\frac{z}{2}\right)^{q+2} C_{-(2L+1)+q+2}(z) - \frac{1}{4(q+3)}\left(\frac{z}{2}\right)^{q+3} C_{-(2L+1)+q+3}(z)\right\} = \left(\frac{z}{2}\right)^{q+4} C_{-(2L+1)+q}(z). \tag{10}$$

Choosing $V_0^{(L)}(z) = J_{-(2L+1)}(z)$, we obtain the series

$$J_{-(2L+1)}^n(z) = \sum_{k=0}^{\infty} n^{-2k} \sum_{q=2k}^{3k} b_{k,q}^{(L)}(z/2)^q J_{-(2L+1)+q}(z), \tag{11}$$

in which the $b_{k,q}^{(L)}$ are determined from (10) in the same manner as the $a_{k,q}^{(L)}$ are obtained from (7), so that $(z/2)^{-(2L+1)}$ occurs only in the term $V_0^{(L)}(z)$. In the following discussion we shall always mean by $a_{k,q}^{(L)}$ and $b_{k,q}^{(L)}$ the coefficients appearing in *these* series, determined as described above. The proof that (11) converges absolutely and uniformly for $|z| \leq R$ and $|n| \geq n_0$ is carried out in a manner identical with the proof of the convergence of (9).

Since both $(z/2)J_{2L+1}^n(z)$ and the regular Whittaker function[8] $M_{n,L+\frac{1}{2}}(z^2/4n)$ satisfy (1), it follows by comparison of the coefficients of $(z/2)^{2L+2}$ in the series expansions of both functions that[5]

$$J_{2L+1}^n(z) = \left[\frac{n^{L+1}}{(z/2)\Gamma(2L+2)}\right] M_{n,L+1/2}(z^2/4n). \tag{12}$$

Similarly, if $(2L + 1)$ is not an integer, we may show that $(z/2)J_{-(2L+1)}^n(z)$ is related to $M_{n,-L-\frac{1}{2}}(z^2/4n)$ by a formula obtained from (12) by replacing L by $-L-1$. Hence if $(2L + 1)$ is not an integer, (9) and (11) are two independent solutions of (1), both of which are satisfactorily convergent in both z and $1/n$. Kuhn has proved the existence of two such series solutions[5] and has exhibited (9); however, instead of (11) he supposed that the series obtained from (9) by replacing $J_m(z)$ by $Y_m(z)$ converged, whereas in fact the resulting series diverges, as we shall show.

If $(2M + 1)$ is a positive integer, the function $M_{n,-M-\frac{1}{2}}(z^2/4n)$ is not defined, and we may show that[9]

$$\lim_{L\to M} J_{-2L-1}^n(z) = \frac{\Gamma(n+M+1)\cos(2M+1)\pi}{n^{2M+1}\Gamma(n-M)} J_{2M+1}^n(z). \tag{13}$$

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[8] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge Univ. Press, 1952, p. 337

[9] G. H. Wannier, Phys. Rev. **64**, 358 (1943)

Hence in order to obtain a second solution of (1) which is independent of $J^n_{2L+1}(z)$ for *all* values of $2L + 1$, we define[9]

$$N^n_{2L+1}(z) = \left[\frac{\Gamma(n + L + 1)}{n^{2L+1}\Gamma(n - L)} J^n_{2L+1}(z) \cos (2L + 1)\pi - J^n_{-2L-1}(z)\right]\left(\frac{1}{\sin (2L + 1)\pi}\right) \tag{14}$$

and understand that, for integral values of $(2L + 1)$, $N^n_{2L+1}(z)$ is defined by the limit of the expression on the right of (14) as $(2L + 1) \to$ integer.

III. Series representations of $N^n_{2L+1}(z)$. To derive a convergent representation of $N^n_{2L+1}(z)$ in a power series in $(1/n^2)$ we may substitute (9) and (11) into (14), since both of these series converge absolutely for all L. However, for non-integral values of $(2L + 1)$ we cannot expand the coefficient of $J^n_{2L+1}(z)$ in (14) in a convergent power series in $(1/n^2)$ because $\Gamma(n + L + 1)$ has a pole whenever $(n + L + 1)$ is a negative integer or zero. We therefore write (14) as

$$\begin{aligned} N^n_{2L+1}(z) \sin (2L + 1)\pi = &\left[\frac{\Gamma(n + L + 1)}{n^{2L+1}\Gamma(n - L)} - 1\right] \cdot \\ &\cdot \sum_{k=0}^{\infty} n^{-2k} \sum_{q=2k}^{3k} a^{(L)}_{k,q}(z/2)^q J_{2L+1+q}(z) \cos (2L + 1)\pi \\ &+ \sum_{k=0}^{\infty} n^{-2k}\left\{\sum_{q=2k}^{3k} a^{(L)}_{k,q}(-)^q(z/2)^q J_{2L+1+q}(z) \cos (2L + 1 + q)\pi \right. \\ &\left. - \sum_{q=2k}^{3k} b^{(L)}_{k,q}(z/2)^q J_{-(2L+1)+q}(z)\right\}. \end{aligned} \tag{15}$$

The $k = 0$ term of the second series in (15) is seen to be $\sin (2L + 1)\pi\ Y_{2L+1}(z)$, since $a^{(L)}_{00} = b^{(L)}_{00} = 1$. Since the first series in (15) is simply a multiple of $J^n_{2L+1}(z)$, and because both $(z/2)J^n_{2L+1}(z)$ and $(z/2)N^n_{2L+1}(z)$ satisfy (1), we see from the form of (15), Eq. (7), and the above value of the $k = 0$ term that the $k = 1$ term of the second series in (15) is determined by (4) to be

$$n^{-2}[a^{(L)}_{1,2}(z/2)^2 Y_{2L+3}(z) + a^{(L)}_{1,3}(z/2)^3 Y_{2L+4}(z)] \sin (2L + 1)\pi \tag{16}$$

plus some linear combination of the solutions of the homogeneous equation (4a), $J_{2L+1}(z)$ and $J_{-2L-1}(z)$. We can determine this latter combination by using the Bessel function recurrence relations to re-express the terms involving $(1/n^2)J_{-2L-1+q}(z)$ in the second series. We see that we shall obtain

$$\begin{aligned} -b^{(L)}_{1,2}(z/2)^2 J_{-2L-1+2}(z) - b^{(L)}_{1,3}(z/2)^3 J_{-2L-1+3}(z) \\ = -a^{(L)}_{1,2}(z/2)^2 J_{-2L-3}(z) + a^{(L)}_{1,3}(z/2)^3 J_{-2L-4}(z) \\ - A(L)J_{-2L-1}(z) - B(L)J_{2L+1}(z). \end{aligned} \tag{17}$$

We need not carry this out explicitly in order to obtain the polynomials $A(L)$ and $B(L)$, however. Because the recurrence relations relate Bessel functions of orders differing by integers, if $2(2L + 1)$ is not an integer[10] we shall not obtain $J_{2L+1}(z)$ in an expression for $J_{-2L-1+q}(z)$, so that $B(L) = 0$. We may obtain $A(L)$ if we notice that when $2L + 1$ equals an integer, $2M + 1$, the left side of (15) vanishes for all n and that

[10]We make use of non-integral values of $2(2L + 1)$ in order to prove $B(L) = 0$. Both sides of (17) being continuous functions of L, the conclusion holds for $2(2L + 1) =$ integer as well.

$$\frac{\Gamma(M+n+1)}{n^{2M+1}\Gamma(n-M)} = \sum_{p=0}^{[M+1]} \frac{b_p(M)}{n^{2p}}, \tag{18}$$

the $b_p(M)$ being polynomials the calculation of which is outlined in Appendix II, and $[M+1]$ denoting the largest integer *less* than $M+1$ (i.e. $[M+1]$ equals M if M is an integer, $(M+\frac{1}{2})$ if M is half an odd integer). Therefore the coefficients of $(1/n^2)$ on the right of (15) must vanish when $L = M$, and since $J_{-2M-1}(z) = (-1)^{2M+1}J_{2M+1}(z)$, we find on combining all terms in $1/n^2$ in (15) when $L = M$ and using (16), (17), and (18), that $A(L)$ must differ from $b_1(L)$ by at most a function of L that vanishes for all M. Since $A(L)$ is itself a polynomial, however, we have that $A(L) = b_1(L)$.

Continuing this argument to higher powers of $(1/n^2)$, we find that the desired form of the series is given us immediately: the term in n^{-2k} is determined from that for $n^{-2(k-1)}$ by the use of the differential equations (4) and (7) except for a multiple of $n^{-2k}J_{-2L-1}(z)$; this multiple is determined from the requirement that this term cancels for all integral values of $2M+1$ the term $-n^{-2k}b_k(M)J_{2M+1}(z)$ in the first series of (15).

We now wish to use this information to rewrite (15) in a form such that when $L = M$ each term of the series vanishes. We can't do this, once and for all, for arbitrary M because, as we shall see, we should have to make use of a divergent asymptotic series which we want to avoid. But if we are interested in a particular M we can conveniently proceed as follows. We know that the series (9) and (11) converge absolutely, so that we can add to the second series in (15) a finite polynomial in powers of $(1/n^2)$ multiplied by (9) and rearrange the resulting combined series in powers of $(1/n^2)$ without disturbing convergence. Choosing this polynomial to be $\sum_{p=1}^{[M+1]} b_p(L)n^{-2p}$, adding it in as described and subtracting it out in combination with the first series, we have finally on making use of the relation $b_0(L) = 1$

$$\begin{aligned} N^n_{2L+1}(z) &= \left[\frac{\Gamma(L+n+1)}{n^{2L+1}\Gamma(n-L)} - \sum_{p=0}^{[M+1]} \frac{b_p(L)}{n^{2p}}\right] \frac{\cos(2L+1)\pi}{\sin(2L+1)\pi} J^n_{2L+1}(z) \\ &\quad + \sum_{k=0}^{[M+1]} n^{-2k} \sum_{p=0}^{k} b_p(L) \sum_{q=2(k-p)}^{3(k-p)} a^{(L)}_{k-p,q}(z/2)^q Y_{2L+1+q}(z) \\ &\quad + \sum_{k=[M+1]+1}^{\infty} n^{-2k}\Bigg\{\sum_{p=0}^{[M+1]} b_p(L) \sum_{q=2(k-p)}^{3(k-p)} a^{(L)}_{k-p,q}(z/2)^q Y_{2L+1+q}(z) \\ &\quad - \sum_{p=[M+1]+1}^{k} \left[\frac{b_p(L)}{\sin(2L+1)\pi}\right] \sum_{q=2(k-p)}^{3(k-p)} a^{(L)}_{k-p,q}(-)^q(z/2)^q J_{-(2L+1+q)}(z)\Bigg\}. \end{aligned} \tag{19}$$

We see that the series is now in a particularly convenient form for taking the limit $L \to M$, where, we recall, $2M+1$ is a particular integer. Thus using (18) we can write for $L \to M$

$$\begin{aligned} G_M(M, n) &= \lim_{L\to M} \frac{\cos(2L+1)\pi}{\sin(2L+1)\pi} \left\{\frac{\Gamma(L+n+1)}{n^{2L+1}\Gamma(n-L)} - \sum_{p=0}^{[M+1]} \frac{b_p(L)}{n^{2p}}\right\} \\ &= \frac{1}{2\pi}\left\{\frac{d}{dL}\left(\frac{\Gamma(n+L+1)}{n^{2L+1}\Gamma(n-L)}\right) - \sum_{p=0}^{[M+1]} n^{-2p}\frac{d}{dL}(b_p(L))\right\}_{L=M} \\ &= \frac{1}{2\pi}\Bigg\{\frac{\Gamma(L+n+1)}{n^{2L+1}\Gamma(n-L)}[\Psi(n+L+1) + \Psi(n-L) - 2\ln(n)] \\ &\qquad - \sum_{p=0}^{[M+1]} \frac{1}{n^{2p}}\frac{d}{dL}(b_p(L))\Bigg\}_{L=M} \end{aligned} \tag{20}$$

where $\Psi(x) = (d/dx) \ln \Gamma(x)$. Also it is in general true that $b_p(M) = 0$ if $p > [M + 1]$. Thus all terms in (19) vary continuously as $L \to M$.

We notice in (19) that $(z/2)N^n_{2L+1}(z)$ has been expressed as a multiple $G_M(L, n)$ of the regular function $(z/2)J^n_{2L+1}(z)$ plus an absolutely convergent series in $(1/n^2)$ which we shall call $(z/2)Q^n_{2L+1}(z, M)$ and which is clearly an irregular solution of (1). Both $(z/2)N^n_{2L+1}(z)$ and $(z/2)Q^n_{2L+1}(z, M)$ satisfy a Wronskian relation with $(z/2)J^n_{2L+1}(z)$:

$$[(z/2)J^n_{2L+1}(z)][(z/2)(d/dz)(z/2)N^n_{2L+1}(z)] - [(z/2)N^n_{2L+1}(z)][(z/2)(d/dz)(z/2)J^n_{2L+1}(z)] = z^2/4\pi. \tag{21}$$

The relation involving $Q^n_{2L+1}(z, M)$ is identical with (21) if N is replaced by Q. A number of other formulas involving $N^n_{2L+1}(z)$ are given in Appendix III.

An asymptotic series for $N^n_{2L+1}(z)$ in powers of $(1/n^2)$ for arbitrary L may be obtained from the convergent representation (19) if we make use of the asymptotic expansion (A3) (Appendix II), valid for $|\arg(n)| \leq \pi - \Delta < \pi$, and rearrange (19), making use of the fact that we may multiply and add asymptotic series. Combining terms in $J_{2L+1+q}(z)$ and $J_{-2L-1-q}(z)$ by means of the definition[11] of the Weber function, we obtain

$$N^n_{2L+1}(z) \underset{|n|\to\infty}{\sim} \sum_{p=0}^{\infty} n^{-2p} b_p(L) \sum_{k=0}^{\infty} n^{-2k} \sum_{q=2k}^{3k} a^{(L)}_{k,q}(z/2)^q Y_{2L+1+q}(z). \tag{22}$$

Multiplying by the reciprocal expansion of (A3), we have also

$$\frac{\Gamma(n-L)n^{2L+1}}{\Gamma(n+L+1)} N^n_{2L+1}(z) \underset{|n|\to\infty}{\sim} \sum_{k=0}^{\infty} n^{-2k} \sum_{q=2k}^{3k} a^{(L)}_{k,q}(z/2)^q Y_{2L+1+q}(z). \tag{23}$$

Except when $(2L + 1)$ equals half an odd integer, both (22) and (23) may be proved to be divergent series[6] which are asymptotic representations in n, for $|\arg(n)| \leq \pi - \Delta < \pi$, of the functions on the left of the equations. We note that (23) has the same form as the convergent series (9) but with $Y_m(z)$ replacing $J_m(z)$.[12]

IV. Acknowledgments. The author would like to thank Professor Harvey Brooks for discussing this work with him and for making several helpful criticisms.

Appendix I. Proof of the convergence of (9). It follows from (7) that the numbers $a^{(L)}_{k,q}$ have an upper bound $\beta^{(L)}$. From the series definition of the Bessel function $J_m(z)$ we may show that

$$|J_m(z)| < (|z|/2)^m (m!)^{-1} \exp(|z|^2/4). \tag{A1}$$

Since $2k \leq q \leq 3k$ in (9), by writing (9) in the form (2) we may show that, if k_0 is sufficiently large, then for all $k \geq k_0$,

$$|{}^0U^{(L)}_{ks}(z)n^{-2k}| < \beta^{(L)} \exp(|z|^2/4)(|z|/2)^{2L+4k+2} |n|^{-2k} [(2L+2k)!]^{-1}. \tag{A2}$$

If we replace $|z|$ by R and $|n|$ by n_0 on the right of (A2), where R and n_0 are arbitrary positive numbers, then the series in k formed from the resulting quantities can be shown

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[11]Whittaker and Watson, *op. cit.* p. 370

[12]The series (23) is one of those given by Kuhn in Footnote 5. We have also verified for $L = 0, 1, 2, 3$ that Kuhn's *alternate* generating procedure with $C_0(z) = Y_0(z)$ and $C_1(z) = Y_1(z)$ yields the asymptotic series (22) for $(z/2)N^n_{2L+1}(z)$ for these values of L, for which $b_p(L) = 0$ when $p \geq L + 1$. Hence Kuhn's tables for the irregular function are convenient if the asymptotic representation is sufficiently accurate.

to converge absolutely by the ratio test. It therefore follows from (A2), the comparison test, and the Weierstrass M-test that the original series (9) converges absolutely and uniformly for $|z| \leq R$, $|n| \geq n_0$.

Appendix II. The polynomials $b_p(M)$. From the exponential form of Stirling's series[13] for $\Gamma(x)$, we may show that for an arbitrary L the following asymptotic relation holds for $|\arg(n)| \leq \pi - \Delta < \pi$ as $|n| \to \infty$:

$$\frac{\Gamma(n+L+1)}{n^{2L+1}\Gamma(n-L)} \sim \sum_{p=0}^{\infty} \frac{b_p(L)}{n^{2p}}, \tag{A3}$$

the $b_p(L)$ being polynomials in L. In order to determine these, we may let L equal an integer M. Then from the relation $\Gamma(x+1) = x\Gamma(x)$ we may evaluate the left side of (A3), obtaining

$$n^{-2M}(n^2 - M^2)(n^2 - (M-1)^2) \cdots (n^2 - 1) = \sum_{p=0}^{M} b_p(M) n^{-2p}. \tag{A4}$$

Clearly $b_p(L)$ vanishes if L equals an integer less than p. It also vanishes if L is a half integer less than $p - 1$. We see from (A4) that $b_p(M)$ for integral M is equal to $(-1)^p$ times the sum of all different products of the squares of p integers less than or equal to M, no two integers in any product being the same. This may be written as

$$b_p(M) = (-)^p \sum_{k_p=p}^{M} k_p^2 \sum_{k_{p-1}=p-1}^{k_p-1} k_{p-1}^2 \sum_{k_{p-2}=p-2}^{k_{p-1}-1} k_{p-2}^2 \cdots \sum_{k_1=1}^{k_2-1} k_1^2. \tag{A5}$$

Therefore

$$b_p(M) = -\sum_{k_p=p}^{M} k_p^2 b_{p-1}(k_p - 1). \tag{A6}$$

The relation (A6), the fact that $b_0(M) = 1$, and the formula

$$\sum_{k=1}^{M} k(k+1)(k+2)\cdots(k+n) = M(M+1)(M+2)\cdots(M+n+1)(n+2)^{-1} \tag{A7}$$

may be used to calculate the $b_p(M)$ as polynomials in M. Since the resulting polynomials are correct for all integral values of $M \geq p$, we can replace M by L in order to obtain the $b_p(L)$ appropriate to (A3) for general values of L. These polynomials are given in the reference of Footnote 6 for $p \leq 7$.

Appendix III. Miscellaneous formulas. The following formulas are useful in manipulating the Coulomb functions. They may be derived from the definitions of the various functions and formulas given in Chapter 16 of Whittaker and Watson[8]. We should note that (A8) corrects an error in the relation of Example 2, p. 346 of that text. Also, (A9) was given incorrectly by Wannier[9] and corrected by Kuhn[5].

$$M_{n,L+1/2}(s) = \frac{\Gamma(2L+2)}{\Gamma(L+1-n)} e^{in\pi} W_{-n,L+1/2}(se^{i\pi}) + \frac{\Gamma(2L+2)}{\Gamma(n+L+1)} e^{i\pi(n-L-1)} W_{n,L+1/2}(s). \tag{A8}$$

$$[(2L+1) \neq \text{negative integer}].$$

[13]Whittaker and Watson, *op. cit.*, p. 253

$$W_{n,L+1/2}(z^2/4n) = [\Gamma(n+L+1)n^{-L-1}(z/2)J^n_{2L+1}(z)\cos(n-L-1)\pi + \Gamma(n-L)n^L(z/2)N^n_{2L+1}(z)\sin(n-L-1)\pi]. \tag{A9}$$

$$(z/2)N^n_{2L+1}(z) = (n^{-L}e^{i\pi n})(1/\pi)[e^{-i\pi(L-1/2)}\sin\pi(n-L)\Gamma(L+1-n)W_{n,L+1/2}(z^2/4n) - \Gamma(L+1+n)\cos\pi(n-L)W_{-n,L+1/2}((z^2/4n)e^{i\pi})]. \tag{A10}$$

$$M_{n,L+1/2}(s) \underset{|s|\to\infty}{\sim} \frac{\Gamma(2L+2)}{\Gamma(L+1-n)}e^{s/2}s^{-n} + \frac{\Gamma(2L+2)e^{i(n-L-1)\pi}}{\Gamma(n+L+1)}e^{-s/2}s^n. \tag{A11}$$

$$[(2L+1) \neq \text{negative integer}; -\pi < \arg(s) < 0.]$$

$$(z/2)N^n_{2L+1}(z) \underset{|z|\to\infty}{\sim} (n^{-L}e^{in\pi})(1/\pi) \cdot[e^{-i\pi(L-1/2)}\sin\pi(n-L)\Gamma(L+1-n)\exp(-z^2/8n)(z^2/4n)^n - \Gamma(L+1+n)\cos\pi(n-L)\exp(z^2/8n)e^{-i\pi n}(z^2/4n)^{-n}]. \tag{A12}$$

$$[-\pi < \arg(z^2/4n) < 0].$$

Appendix IV. Derivation of formulas for the repulsive field. Equations (9), (19), and (23) are valid for arbitrary complex values of the parameters L, z, and n. From (1) we see for attractive fields that if we take r and z positive real, then real (and for convenience, positive) values of n correspond to negative energies, pure imaginary values of n to positive energies. Again from (1), we see that for repulsive fields and positive energies we may conveniently choose r to be real and negative and n and z to be negative pure imaginary numbers. We shall restrict our attention to values of L that are positive integers or zero.

The regular and irregular functions, F_L and G_L respectively, used by Breit and others[1-3], have the asymptotic forms for fixed energy and large radius

$$F_L \underset{\rho\to\infty}{\sim} \sin[\rho - (\pi L/2) - \eta\ln 2\rho + \arg\Gamma(L+1+i\eta)],$$
$$G_L \underset{\rho\to\infty}{\sim} \cos[\rho - (\pi L/2) - \eta\ln 2\rho + \arg\Gamma(L+1+i\eta)]. \tag{A13}$$

Here in Breit's notation $\rho = |r|/|n|$, $\eta = |n|$, and $x = |z| = (8\rho\eta)^{1/2}$. Comparing these with the asymptotic expansions of our functions for large $|r|$, got from (A11) and (A12), we find that

$$F_L = \frac{|\Gamma(L+1+i\eta)|e^{-\pi\eta/2}e^{i\pi(L+1)}}{2i\eta^{L+1}}\left(\frac{x}{2}\right)J^{-i\eta}_{2L+1}(-ix) \tag{A14}$$

and that

$$\left(\frac{-ix}{2}\right)N^{-i\eta}_{2L+1}(-ix) = |\Gamma(L+1+i\eta)|\eta^{-L}e^{\pi\eta/2}(1/\pi)[e^{\pi\eta-\pi iL}(iF_L) - e^{-\pi\eta+\pi iL}G_L]. \tag{A15}$$

From this latter formula we see that since both F_L and G_L are real, then

$$G_L = \frac{(-1)^{L+1}e^{\pi\eta/2}\eta^L\pi}{|\Gamma(L+1+i\eta)|}\text{ R.P.}\left\{\left(\frac{-ix}{2}\right)N^{-i\eta}_{2L+1}(-ix)\right\}. \tag{A16}$$

We can obtain the real part (R.P.) of $(-ix/2)N_{2L+1}^{-i\eta}(-ix)$ from either (19) or (23) if we make use of the formulas[14]

$$\begin{aligned}(-ix/2)^m J_m(-ix) &= e^{-\pi i m}(x/2)^m I_m(x),\\ (-ix/2)^m Y_m(-ix) &= (-2/\pi \cos \pi m)(x/2)^m K_m(x) - i\, e^{-\pi i m}(x/2)^m I_m(x).\end{aligned} \tag{A17}$$

Using (23), we evidently arrive at the asymptotic expansion of G_L given by Breit and Hull[1], and from (19) we obtain the convergent series of Jackson and Blatt[3]. We have explicitly checked the terms containing $(1/\eta^2)$ to the powers zero and one in our resulting expansions for $L = 0$ against those given by Breit and Bouricius[1] and Jackson and Blatt. The agreement is complete except that the latter authors have omitted a minus sign in the term $H_1(r)$ of their equation (A3.14); the sign of this term is given correctly by Breit and Bouricius.

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[14]*Ibid.*, pp. 372-373

BOOK REVIEWS

Abwickelbare Flächen. By Hans Schmidbauer. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. V + 66 pp. $1.57.

The profusely illustrated booklet treats the elementary descriptive geometry of developable surfaces with particular attention to problems of engineering design.

W. Prager

Relativity: the special theory. By J. L. Synge. North Holland Publishing Co., Amsterdam, and Interscience Publishers, Inc., New York, 1956, xiv + 450 pp. $10.50.

The special theory of relativity is now over fifty years old, and it is here treated as an established theory that needs no special pleading for its acceptance. Professor Synge spends no time on the usual preliminary discussions of ether, absolute motion, Einstein's operational definition of simultaneity, and the like. He begins with the idea that the physical world can be pictured in terms of a four-dimensional metrical continuum, and on the basis of quite mild assumptions about the temporal ordering of events shows that the local metric must be of the Minkowskian type, the Newtonian type appearing as a limiting case. He then makes the specific assumption that the space-time is flat and proceeds to develop the special theory deductively, pausing at frequent intervals to make contact with the familiar, though not wholly appropriate concepts of Newtonian physics, but returning always to the Olympian viewpoint of Minkowski. The first three chapters set the Minkowskian stage. There follow two chapters on the Lorentz transformation, the first discussing it mathematically in considerable detail, and the second applying it to derive such familiar results as the contraction of lengths, and also to construct a special relativity model of an expanding universe. Subsequent chapters take up the mechanics of a particle and collision problems, the mechanics of discrete systems, the mechanics of a continuum, the electromagnetic field in vacuo, and fields and charges. There are five appendices discussing various technical points, a list of references, and an excellent index.

Of the wealth of interesting items in this book only a few can be mentioned here. Thus the author emphasizes the relativistic sameness of photons, and comes to the logical, if at first startling, conclusion that *luminosity* should be defined in terms of all photons and not just those of visible light in a given frame. Again, he stresses the difficulties attending the definition of a rigid body in relativity, and avoids having to face them by postulating an ingenious mechanism for the transmission of impulses between particles. He shows how center of mass, a by no means obvious concept in a theory that lacks absolute simultaneity, can be defined with the aid of angular momentum. He shows how conservation of angular momentum can ameliorate an awkward and surprising lack of uniqueness in certain collision problems. He gives detailed discussions of many collisions involving material particles and photons that are of special interest to nuclear physicists, contrasting the various relative, three-dimensional pictures with their absolute, four-dimensional counterparts, though here he may confuse the reader by his failure to distinguish between space-time diagrams of world lines and those of vectors. He makes the treatment of the Compton effect particularly illuminating by examining it in four different frames of reference. Out of the pure electromagnetic field he constructs a remarkable non-singular model of a material particle which, because it is free of singularities, must unfortunately be uncharged. He makes every occasion an opportunity for four-dimensional geometrical elucidation. And he does many other things of which the above is a wholly inadequate sample.

Despite its 450 pages, the book confines itself to what may be called the classical as distinguished from the quantum aspects of the special theory of relativity. Though spinors are not wholly absent, they enter only briefly in connection with the Lorentz transformation.

While the above may indicate the scope and nature of the book, it does not convey the author's characteristic blend of insight and gentle wit that gives the book its special quality. The explanations are patient and penetrating, and remarkably lucid, often being expressed in the scientific equivalent of words of one syllable. In the preface the author writes that his aim is "to make space-time a real workshop for physicists, and not a museum visited occasionally with a feeling of awe." In this he has admirably succeeded.

Banesh Hoffmann

(Continued on p. 64)

THE EFFECT OF TRANSVERSE SHEAR DEFORMATION ON THE BENDING OF ELASTIC SHELLS OF REVOLUTION*

BY

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1. Introduction. The classical theory of thin elastic shells of revolution with small axisymmetric displacements, due to H. Reissner [1] for spherical shells of uniform thickness, and Meissner [2, 3] for the general shells of revolution, was recently reconsidered by E. Reissner [4], where reference to the historical development of the subject may be found. Although the formulation of the linear theory and the resulting differential equations contained in [4] differ only slightly from those of H. Reissner and Meissner, they offer certain advantages not revealed in earlier formulations.

As has been recently pointed out, the improvement of the linear theory of thin shells, by inclusion of the effects of both transverse shear deformation and normal stress, requires the formulation of suitable stress strain relations and appropriate boundary conditions which, for shells of uniform thickness, have been very recently carried out by E. Reissner [5] and the present author [6]**. The latter also contains explicit stress strain relations when only the effect of transverse shear deformation is fully accounted for but that of normal stress is neglected.

The present paper is concerned with the small axisymmetric deformation of elastic shells of revolution, where only the effect of transverse shear deformation is retained. The basic equations which include the appropriate expression for the transverse (shear) stress resultant due to the variation in thickness, are reduced to two simultaneous second-order differential equations in two suitable dependent variables. These equations are then combined into a single complex differential equation which is valid for shells of uniform thickness, as well as for a large class of variable thickness. Finally, by an extension of the method of asymptotic integration due to Langer [8], the general solution of the complex differential equation is discussed.

2. The basic equations in surface-of-revolution coordinates. The parametric equations of the middle surface of the shell may be written as

$$r = r(\xi), \qquad z = z(\xi), \tag{2.1}$$

where ξ, together with the polar angle θ in the x, y-plane constitute the coordinates of the middle surface. Denoting by ϕ the inclination of the tangent to the meridian of the shell, then

$$\tan \phi = \frac{dz}{dr} \tag{2.2}$$

and

$$r' = \alpha \cos \phi, \qquad z' = \alpha \sin \phi, \tag{2.3}$$

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**The stress strain relations derived in both [5] and [6] were obtained by application of E. Reissner's variational theorem [7]. References to earlier work on the subject appear in [5] and [6].

where

$$\alpha = [(r')^2 + (z')^2]^{1/2} \tag{2.4}$$

and prime denotes differentiation with respect to ξ.

The square of the linear element for the triply orthogonal curvilinear coordinate system ξ, θ, and ζ (measured along the outward normal to the middle surface) is

$$ds^2 = \alpha^2\left(1 + \frac{\zeta}{R_\xi}\right)^2 d\xi^2 + r^2\left(1 + \frac{\zeta}{R_\theta}\right)^2 d\theta^2 + d\zeta^2, \tag{2.5}$$

where

$$R_\xi = -\frac{\alpha}{\phi'}, \qquad R_\theta = -\frac{r}{\sin\phi} \tag{2.6}$$

are the principal radii of the curvature of the middle surface.

For axisymmetric deformation of shells of revolution, the displacements in the tangential and normal directions may be taken in the form:

$$U_\xi = u(\xi) + \zeta\beta(\xi), \qquad W = w(\xi), \tag{2.7}$$

where u and w denote the components of displacements at the middle surface, and β is the change of the slope of the normal to the middle surface of the shell. As pointed out in both [5] and [6], the approximation (2.7) for the displacements is consistent with the neglect of the effect of transverse normal stress in the stress strain relations for a thin shell. It is convenient to express the displacements u and w in terms of the corresponding components u_r and w_z (on the middle surface $\zeta = 0$) along the radial and axial directions, respectively. Thus,

$$u_r = u\cos\phi - w\sin\phi, \qquad w_z = u\sin\phi + w\cos\phi \tag{2.8a}$$

or alternatively,

$$u = u_r\cos\phi + w_z\sin\phi, \qquad w = w_z\cos\phi - u_r\sin\phi. \tag{2.8b}$$

The components of strain, as given in [5, 6], with the aid of (2.8) may be written in the form

$$\epsilon_\xi^0 = \frac{u_r' + z'\omega}{r'}, \qquad \epsilon_\theta^0 = \frac{u_r}{r},$$
$$\gamma_{\xi\zeta}^0 = w_z'\frac{\cos\phi}{\alpha} - u_r'\frac{\sin\phi}{\alpha} + \beta, \tag{2.9a}$$

$$\kappa_\xi = \frac{\beta'}{\alpha}, \qquad \kappa_\theta = \frac{r'}{r}\frac{\beta}{\alpha} \tag{2.9b}$$

and the relevant compatibility equation, obtained from (2.9a) by elimination of u_r is

$$(r'\epsilon_\xi^0) = (r\epsilon_\theta^0)' + z'\omega, \tag{2.10a}$$

where

$$\omega = \gamma_{\xi\zeta}^0 - \beta. \tag{2.10b}$$

It may be noted that, upon the neglect of terms involving $\gamma_{\xi\zeta}^0$ (which represent the effect of transverse shear deformation), (2.9) and (2.10) reduce to the corresponding expressions of the classical theory.

The stress differential equations of equilibrium, given in [4], may be written as

$$
\begin{aligned}
(rP_V)' &= -r\alpha q_V , \\
(rP_H)' - \alpha N_\theta + r\alpha q_H &= 0, \qquad (2.11)\\
(rM_\xi)' - \alpha \cos\phi M_\theta - r\alpha(P_V \cos\phi - P_H \sin\phi) &= 0,
\end{aligned}
$$

where P_H and P_V are "horizontal" and "vertical" stress resultants, respectively; q_H and q_V are the components of the applied load in the horizontal and vertical directions; M_ξ and M_θ denote the stress couples; and the stress resultants N_ξ and N_θ, as well as the transverse stress resultant V (due to $\tau_{\xi\zeta}$), are related to P_H and P_V by

$$
\begin{aligned}
\alpha N_\xi &= r'P_H + z'P_V , \\
\alpha V &= -z'P_H + r'P_V , \qquad (2.12)\\
\alpha N_\theta &= (rP_H)' + r\alpha q_H .
\end{aligned}
$$

We recall that the stress strain relations (as well as the stress differential equations of equilibrium) which are employed in the classical theory of shells (and plates) of variable thickness, are those for shells of uniform thickness, although the resulting differential equations take into account the effect of thickness variation. As the present analysis is concerned mainly with the effect of transverse shear deformation, it becomes necessary to modify slightly the available stress strain relations of uniform shells [5, 6] by incorporating the effect of thickness variation into the expression for the transverse shear stress $\tau_{\xi\zeta}$. To this end, we replace in Eqs. (2.9) of Ref. [6] the expression corresponding to $\tau_{\xi\zeta}$ by

$$
\left(1 + \frac{\zeta}{R_\theta}\right)\tau_{\xi\zeta} = \frac{3}{2h} V\left[1 - \left(\frac{\zeta}{h/2}\right)^2\right] - 3\left(\frac{h'}{2\alpha}\right)\left(\frac{M_\xi}{h^2}\right)\left[1 - 3\left(\frac{\zeta}{h/2}\right)^2\right] + \left(\frac{h'}{2\alpha}\right)\left(\frac{N_\xi}{h}\right)\left(\frac{\zeta}{h/2}\right), \qquad (2.13)
$$

which, together with σ_ξ, satisfies the stress boundary conditions on $\zeta = \pm h/2$, where the direction cosines of the outward normal to the middle surface are $\{-h'/2\alpha, 0, 1\}/[1 + (h'/2\alpha)^2]^{\frac{1}{2}}$. While expression (2.13) will not change the differential equations of equilibrium (2.11), it does contribute to the stress strain relations, as may be seen from the variational equation employed in [5, 6]. However, in view of the neglect of the transverse normal stress, together with neglect of second-order corrections in h/R in comparison with first-order corrections, the resulting stress strain relations for a uniform shell (in the form of (3.6) of Ref. [6]), except for a modification in the transverse shear stress strain relation, remain unaltered.

We close this section by recording the stress strain relations in a manner suitable for subsequent analysis. Thus,

$$
\begin{aligned}
\epsilon_\xi^0 &= \frac{1}{C}(N_\xi - \nu N_\theta) + k\lambda\left[\frac{1}{\alpha}\left(\beta' + \nu\frac{r'}{r}\beta\right)\right], \\
\epsilon_\theta^0 &= \frac{1}{C}(N_\theta - \nu N_\xi) - k\lambda\left[\frac{1}{\alpha}\left(\frac{r'}{r}\beta + \nu\beta'\right)\right], \qquad (2.14a)
\end{aligned}
$$

$$\begin{aligned}
\gamma^0_{\xi\zeta} &= \frac{6}{5Gh}\left[V - \frac{1}{2}\left(\frac{h'}{h}\right)\frac{M_\xi}{\alpha}\right], \\
\alpha M_\xi &= D\left[\left(\beta' + \nu\frac{r'}{r}\beta\right) - \alpha\lambda\left(\frac{u_r' + z'\omega}{r'}\right)\right], \\
\alpha M_\theta &= D\left[\left(\frac{r'}{r}\beta + \nu\beta'\right) + \alpha\lambda\left(\frac{u_r}{r}\right)\right],
\end{aligned} \tag{2.14b}$$

where

$$C = Eh, \qquad G = \frac{E}{2(1+\nu)}, \qquad D = \frac{Eh^3}{12(1-\nu^2)},$$

E and ν are Young's modulus and Poisson's ratio, respectively, and

$$k = \frac{h^2}{12(1-\nu^2)}, \qquad \lambda = \left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right) = \frac{1}{\alpha}\left(\frac{z'}{r} - \phi'\right). \tag{2.15}$$

3. Differential equations of shells of revolution. With β and (rP_H) as basic dependent variables, proper elimination between Eqs. (2.12), (2.14), (2.11), and (2.10) leads to the following two second-order differential equations:

$$\begin{aligned}
&\beta'' + \left[(1 - k\lambda^2)\frac{(rD/\alpha)'}{(rD/\alpha)} - k\lambda\left(2\lambda' + \frac{k'}{k}\lambda\right)\right]\beta' \\
&\quad - \left\{(1 - k\lambda^2)\left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right] + \nu k\lambda\left(2\lambda' + \frac{k'}{k}\lambda\right)\left(\frac{r'}{r}\right)\right\}\beta \\
&\quad + \frac{z'}{(rD/\alpha)}(rP_H) + \left(\frac{\lambda}{C}\right)\left\{-[(r'P_H)' - \nu(rP_H)'']\right. \\
&\quad - \left[\frac{(\lambda/C)'}{(\lambda/C)} + \frac{(rD/\alpha)'}{(rD/\alpha)} - \nu\left(\frac{r'}{r}\right)\right](r'P_H) \\
&\quad \left. + \left[\nu\frac{(\lambda/C)'}{(\lambda/C)} + \nu\frac{(rD/\alpha)'}{(rD/\alpha)} + \left(\frac{r'}{r}\right)\right](rP_H)'\right\} \\
&= \frac{r'}{(rD/\alpha)}(rP_V) + \left(\frac{\lambda}{C}\right)\left\{[(z'P_V)' - \nu(r\alpha q_H)']\right. \\
&\quad + \left[\frac{(\lambda/C)'}{(\lambda/C)} + \frac{(rD/\alpha)'}{(rD/\alpha)} - \nu\left(\frac{r'}{r}\right)\right](z'P_V) \\
&\quad \left. - \left[\nu\frac{(\lambda/C)'}{(\lambda/C)} + \nu\frac{(rD/\alpha)'}{(rD/\alpha)} - \left(\frac{r'}{r}\right)\right](r\alpha q_H)\right\},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&(rP_H)'' + \frac{(r/\alpha C)'}{(r/\alpha C)}(rP_H)' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha C)'}{(r/\alpha C)}\right](rP_H) \\
&\quad - \frac{z'}{(r/\alpha C)}\beta - \frac{k\lambda}{(r/\alpha C)}\left\{\nu\left(\frac{r}{\alpha}\right)\beta'' + \left[2\left(\frac{r'}{\alpha}\right) + \nu\left(\frac{r}{\alpha}\right)' + \nu\left(\frac{r}{\alpha}\right)\frac{(k\lambda)'}{(k\lambda)}\right]\beta'\right. \\
&\quad \left. + \left[\left(\frac{r}{\alpha}\right)\left(\frac{r'}{r}\right)' + \left(\frac{r}{\alpha}\right)'\left(\frac{r'}{r}\right) + \nu\left(\frac{r'}{\alpha}\right)\left(\frac{r'}{r}\right) + \left(\frac{r'}{\alpha}\right)\frac{(k\lambda)'}{(k\lambda)}\right]\beta\right\} \\
&= \left[\frac{r'z'}{r^2} + \nu\frac{(z'/\alpha C)'}{(r/\alpha C)}\right](rP_V) + \nu\frac{z'}{r}(rP_V)' \\
&\quad - \left[\frac{(r/\alpha C)'}{(r/\alpha C)} + \nu\frac{r'}{r}\right](r\alpha q_H) - (r\alpha q_H)' - \frac{z'}{(r/\alpha C)}\gamma^0_{\xi\zeta},
\end{aligned} \tag{3.2}$$

where k and λ are given by (2.15), and $(rP_V) = -\int r\,\alpha q_V\,d\xi$. With β and (rP_H) known, the radial and axial displacements may be determined from the second of (2.14a) and the equation

$$w_s = \int (z'\epsilon_\xi^0 + r'\omega)\,d\xi \tag{3.3}$$

which, with the aid of (2.3), is deduced from (2.9a).

When the effect of transverse shear deformation is neglected (by setting $\lambda = \gamma_{\xi\zeta}^0 = 0$), (3.1) and (3.2) reduce to the corresponding equations of the classical theory given by E. Reissner [4], and which were first derived in a slightly different form by H. Reissner [1] and Meissner [2, 3].

Introduction of the function ψ defined by

$$\psi = \frac{m}{Eh^2}(rP_H), \qquad m = [12(1-\nu^2)]^{1/2}, \tag{3.4}$$

into (3.1) and (3.2) and some rearrangement of terms results in*

$$\begin{aligned}
\beta'' &+ \left[(1 - k\lambda^2)\left(\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right) - 2k\lambda^2\left(\frac{\lambda'}{\lambda} + \frac{h'}{h}\right)\right]\beta' \\
&\qquad - (1 - k\lambda^2)\left[\left(\frac{r'}{r}\right)^2 - \nu\left(\frac{(r'/\alpha)'}{(r/\alpha)} + 3\frac{r'h'}{rh}\right)\right]\beta - \frac{\alpha^2 m}{R_\theta h_0}\left(\frac{h_0}{h}\right)\psi \\
&\qquad + \nu k^{1/2}\lambda\left\{\psi'' + \left[\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)} - 2\nu\frac{r'}{r} + 6\frac{h'}{h}\right]\psi'\right. \\
&\qquad - \frac{1}{\nu}\left[\left(\frac{r'}{r}\right)' + \frac{r'}{r}\left(\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)}\right)\right. \\
&\qquad \left.\left. - \left(\frac{r'}{r}\right)^2 - 2\nu\frac{h''}{h} - 2\nu\frac{h'}{h}\left(3\frac{h'}{h} - \frac{1}{\nu}\frac{r'}{r} + \frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)}\right)\right]\psi\right\} \\
&= F_1,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\psi'' &+ \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h} - \nu[[k^{1/2}\lambda T]]\right]\psi' \\
&\qquad - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha)'}{(r/\alpha)} + \frac{12(1+\nu)}{5}\left(\frac{z'}{r}\right)^2 - 2\frac{h''}{h} - 2\frac{h'}{h}\left(\frac{(r/\alpha)'}{(r/\alpha)} + \frac{\nu}{2}\frac{r'}{r}\right)\right. \\
&\qquad \left. - [[k^{1/2}\lambda T\left(\frac{r'}{r} - 2\nu\frac{h'}{h}\right)]]\right]\psi + \frac{\alpha^2 m}{R_\theta h_0}\left(\frac{h_0}{h}\right)\beta \\
&\qquad - \nu k^{1/2}\lambda\left\{\beta'' + \left[\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)} + \frac{2}{\nu}\frac{r'}{r} + 2\frac{h'}{h} + [[\frac{m}{\nu\lambda h}(1 - k\lambda^2)T]]\right]\beta'\right. \\
&\qquad + \frac{1}{\nu}\left[\left(\frac{r'}{r}\right)' + \frac{r'}{r}\left(\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)}\right) + \nu\left(\frac{r'}{r}\right)^2 + 2\frac{r'h'}{rh}\right. \\
&\qquad \left.\left. + [[\frac{m}{\lambda h}\left(\frac{r'}{r}\right)(1 - k\lambda^2)T]]\right]\beta\right\} \\
&= F_2,
\end{aligned} \tag{3.6}$$

---

*In Eqs. (3.5) and (3.6), as well as in the subsequent analysis, the effect of thickness variation due to shear stress strain relation is placed in double-squared brackets, i.e., [[ ]].

where h_0 is the value of the thickness h at some reference section (say $\xi = \xi_0$); F_1 denotes the right-hand side of (3.1); and F_2 and T are given by

$$F_2 = \frac{m}{Eh^2}\left\{\left[\frac{r'z'}{r^2}\left(1 - \frac{12(1+\nu)}{5}\right) + \nu\frac{(z'/\alpha C)'}{(r/\alpha C)} + [[\frac{\alpha}{R_\theta}k^{1/2}\lambda T]]\right](rP_V)\right.$$
$$\left. + \nu\frac{z'}{r}(rP_V)' - \left[\nu\frac{r'}{r} + \frac{(r/\alpha C)'}{(r/\alpha C)} - \nu[[k^{1/2}\lambda T]]\right](r\alpha q_H)_\xi - (r\alpha q_H)'\right\}, \tag{3.7a}$$

$$T = -\frac{1}{2}\left(\frac{h}{R_\theta}\right)\left(\frac{h'}{h}\right)\left[\frac{12(1+\nu)}{5m}\right]. \tag{3.7b}$$

Since $h/R \ll 1$, then

$$k\lambda^2 = \frac{h^2}{m^2}\left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right) \ll 1;$$

hence, in what follows, terms of the order $O(k\lambda^2)$ will be neglected in comparison with unity. With

$$X_1 = \beta + \nu k^{1/2}\lambda\psi, \qquad X_2 = \psi - \nu k^{1/2}\lambda\beta, \tag{3.8a}$$

from which

$$\beta = (X_1 - \nu k^{1/2}\lambda X_2) + O(k\lambda^2),$$
$$\psi = (X_2 + \nu k^{1/2}\lambda X_1) + O(k\lambda^2), \tag{3.8b}$$

and with the aid of the relations

$$\frac{(k^{1/2}\lambda)'}{(k^{1/2}\lambda)} = \frac{\lambda'}{\lambda} + \frac{h'}{h},$$
$$\frac{(k^{1/2}\lambda)''}{(k^{1/2}\lambda)} = \frac{\lambda''}{\lambda} + 2\frac{\lambda'h'}{\lambda h} + \frac{h''}{h}, \tag{3.9}$$

(3.5) and (3.6) may be written as

$$L(X_1) + \mu^2 f\left(\frac{h_0}{h}\right)\left[1 + \nu k^{1/2}\lambda(g_1 + G_1)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]X_2$$
$$- \nu k^{1/2}\lambda\left(\frac{2}{\nu}\frac{r'}{r} + \frac{\lambda'}{\lambda} - \frac{h'}{h}\right)X_2' = F_1, \tag{3.10}$$

$$L(X_2) - (\gamma - \kappa)X_2$$
$$- \mu^2 f\left(\frac{h_0}{h}\right)\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]X_1$$
$$- \left[\nu k^{1/2}\lambda\left(\frac{2}{\nu}\frac{r'}{r} - \frac{\lambda'}{\lambda} - 3\frac{h'}{h}\right) + [[T]]\right]X_1' = F_2, \tag{3.11}$$

where

$$L(\;) \equiv (\;)'' + \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right](\;)'$$
$$- \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha)'}{(r/\alpha)} - \nu k^{1/2}\lambda\mu^2 f\left(\frac{h_0}{h}\right) - 3\nu\frac{r'h'}{rh}\right](\;),$$

$$g_1 = \frac{1+\nu}{\nu}\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha)'}{(r/\alpha)} - \frac{1}{\nu}\left(\frac{r'}{r}\right)' - \frac{1}{\nu}\left(\frac{r'}{r}\right)\frac{(r/\alpha)'}{(r/\alpha)} - \frac{\lambda''}{\lambda} - \frac{\lambda'}{\lambda}\left[\frac{(r/\alpha)'}{(r/\alpha)} + \frac{1}{\nu}\frac{r'}{r}\right],$$

$$g_2 = 2\left(\frac{r'}{r}\right)^2 + \frac{1}{\nu}\left(\frac{r'}{r}\right)' + \nu\frac{(r'/\alpha)'}{(r/\alpha)} + \frac{12(1+\nu)}{5}\left(\frac{z'}{r}\right)^2 + \frac{1}{\nu}\left(\frac{r'}{r}\right)\frac{(r/\alpha)'}{(r/\alpha)}$$
$$- \frac{\lambda''}{\lambda} - \frac{\lambda'}{\lambda}\left[\frac{(r/\alpha)'}{(r/\alpha)} - \frac{1}{\nu}\frac{r'}{r}\right],$$

$$\begin{aligned}
G_1 &= \frac{h''}{h} + \frac{h'}{h}\left[4\frac{h'}{h} - 3\frac{\lambda'}{\lambda} + \frac{(r/\alpha)'}{(r/\alpha)} - \frac{4+3\nu^2}{\nu}\left(\frac{r'}{r}\right)\right],\\
G_2 &= -3\frac{h''}{h} - \frac{h'}{h}\left[3\frac{h'}{h} + 3\frac{(r/\alpha)'}{(r/\alpha)} + 5\frac{\lambda'}{\lambda} - \frac{2-\nu^2}{\nu}\left(\frac{r'}{r}\right)\right],\\
\gamma &= 2\left[\nu\frac{(r'/\alpha)'}{(r/\alpha)} + \frac{6(1+\nu)}{5}\left(\frac{z'}{r}\right)^2\right],\\
\kappa &= 2\left[\frac{h''}{h} + \frac{h'}{h}\left(\frac{(r/\alpha)'}{(r/\alpha)} - \nu\frac{r'}{r}\right)\right],\\
\mu^2 f(\xi) &= -\frac{\alpha^2 m}{R_\theta h_0},
\end{aligned} \tag{3.12}$$

and it is to be noted that in the last of (3.12), μ is a constant and $f(\xi)$ is independent of thickness $h(\xi)$.

Multiplying (3.11) by $i\delta$ (where δ is a function of ξ), adding the resulting equation to (3.10), and observing that

$$X_1' = (X_1 + i\delta X_2)' - i\delta X_2' - i\delta' X_2, \qquad i = (-1)^{1/2}, \tag{3.13}$$

will result in a fairly intricate complex differential equation. By taking δ in the form

$$\begin{aligned}
\delta\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]
&= -\frac{1}{2}i(\gamma - \kappa)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\\
&\quad + \left\{\left[1 + \nu k^{1/2}\lambda(g_1 + G_1)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]\right.\\
&\quad \cdot\left[1 + \left(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\right)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]\\
&\quad \left. - \left[\frac{1}{2}(\gamma - \kappa)\left(\mu^2 f\frac{h_0}{h}\right)^{-1}\right]^2\right\}^{1/2}
\end{aligned} \tag{3.14}$$

with the restriction that

$$\delta' = \delta'' = 0, \tag{3.15}$$

the complex differential equation mentioned may be reduced to

$$
\begin{aligned}
L(X_1 + i\delta X_2) &+ i\delta\Big[\nu k^{1/2}\lambda\Big(\frac{(k^{1/2}\lambda)'}{(k^{1/2}\lambda)} - \frac{2}{\nu}\frac{r'}{r} + 2\frac{h'}{h}\Big) - [[T]]\Big](X_1 + i\delta X_2)' \\
&- i\delta\mu^2 f\Big(\frac{h_0}{h}\Big)\Big[1 + \Big(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\Big)\Big(\mu^2 f\frac{h_0}{h}\Big)^{-1}\Big](X_1 + i\delta X_2) \\
&= F_1 + i\delta F_2 \\
&+ \Big[\nu k^{1/2}\lambda\Big(\frac{2}{\nu}\frac{r'}{r}(1 + \delta^2) + \frac{\lambda'}{\lambda}(1 - \delta^2) - \frac{h'}{h}(1 + 3\delta^2)\Big) + [[\delta^2 T]]\Big]X_2' .
\end{aligned}
\tag{3.16}
$$

The coefficient of $(X_1 + i\delta X_2)'$ in the above equation (say B) is

$$
B = \frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h} + i\delta\Big[\nu k^{1/2}\lambda\Big(\frac{(k^{1/2}\lambda)'}{(k^{1/2}\lambda)} - \frac{2}{\nu}\frac{r'}{r} + 2\frac{h'}{h}\Big) - [[T]]\Big] \tag{3.17}
$$

and since

$$
\begin{aligned}
\Theta = \exp\Big\{\frac{1}{2}\int B\,d\xi\Big\} = \Big(\frac{r}{\alpha}\Big)^{1/2} h^{3/2} \exp\Big\{\frac{1}{2}\,i\delta(\nu k^{1/2}\lambda) \\
- i\delta\nu\int\Big[k^{1/2}\lambda\Big(\frac{1}{\nu}\frac{r'}{r} - \frac{h'}{h}\Big) + [[\frac{1}{2\nu}T]]\Big]\,d\xi\Big\}
\end{aligned}
\tag{3.18}
$$

then, by means of the transformation

$$
Y = \Theta(X_1 + i\delta X_2) \tag{3.19}
$$

we finally obtain

$$
Y'' + [i^3\mu^2\Psi^2(\xi) + \Lambda(\xi)]Y - \Theta\Phi X_2' = \Theta(F_1 + i\delta F_2) \tag{3.20}
$$

which, in view of the presence of the function X_2' on the left-hand side, may be called the "quasi-normal" form of (3.13). In (3.20), the functions Φ, Ψ^2, and Λ are given by

$$
\Phi(\xi) = (\nu k^{1/2}\lambda)\Big[\frac{2}{\nu}\frac{r'}{r}(1 + \delta^2) + \frac{\lambda'}{\lambda}(1 - \delta^2) - \frac{h'}{h}(1 + 3\delta^2)\Big] + [[\delta^2 T]] \tag{3.21}
$$

and

$$
\begin{aligned}
\Psi^2(\xi) = \Big\{\delta\Big[1 + \Big(\nu k^{1/2}\lambda(g_2 + G_2) - [[\frac{r'}{r}T]]\Big)\Big(\mu^2 f\frac{h_0}{h}\Big)^{-1}\Big] \\
+ \frac{1}{2}\,i\gamma\Big(\mu^2 f\frac{h_0}{h}\Big)^{-1} + i\nu k^{1/2}\lambda\Big\}\Big(\frac{h_0}{h}\Big)f
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\Lambda(\xi) = &-\Big(\frac{r'}{r}\Big)^2 - \frac{1}{2}\frac{(r/\alpha)''}{(r/\alpha)} + \frac{1}{4}\Big[\frac{(r/\alpha)'}{(r/\alpha)}\Big]^2 - \frac{6(1+\nu)}{5}\Big(\frac{z'}{r}\Big)^2 \\
&- \frac{3}{2}\frac{h''}{h} - \frac{3}{4}\Big(\frac{h'}{h}\Big)^2 - \frac{3}{2}\frac{(r/\alpha)'}{(r/\alpha)}\frac{h'}{h} + 3\nu\frac{r'h'}{rh} \\
&+ i\tilde{\delta}\nu k^{1/2}\lambda\Big\{\frac{1}{\nu}\Big(\frac{r'}{r}\Big)' + \frac{1}{\nu}\Big(\frac{r'}{r}\Big)\frac{(r/\alpha)'}{(r/\alpha)} - \frac{1}{2}\frac{\lambda''}{\lambda} + \frac{\lambda'}{\lambda}\Big(\frac{1}{\nu}\frac{r'}{r} - \frac{1}{2}\frac{(r/\alpha)'}{(r/\alpha)}\Big) \\
&- \frac{5}{2}\frac{h''}{h} + \frac{h'}{h}\Big[\frac{4}{\nu}\frac{r'}{r} - \frac{5}{2}\frac{(r/\alpha)'}{(r/\alpha)} - \frac{9}{2}\frac{\lambda'}{\lambda} - \frac{15}{2}\frac{h'}{h}\Big]\Big\}.
\end{aligned}
\tag{3.23}
$$

4. Approximation of δ. Since δ, as given by (3.14) is fairly complicated, we impose a further restriction on the order of magnitude of $(\mu^2)^{-1}$, i.e.,

$$(\mu^2)^{-1} = O(k^{1/2}\lambda). \tag{4.1}$$

Thus, by (4.1), $\nu k^{\frac{1}{2}}\lambda\mu^{-2} = O(k\lambda^2)$ which, as in the previous section, may be neglected in comparison with unity. With this approximation, (3.14) simplifies considerably and reads as follows:

$$\delta = -\frac{1}{2}\,i(\gamma - \kappa)\left(\mu^2 f\,\frac{h_0}{h}\right)^{-1} + \left\{1 - \left[\frac{1}{2}(\gamma - \kappa)\left(\mu^2 f\,\frac{h_0}{h}\right)^{-1}\right]^2\right\}^{1/2}. \tag{4.2}$$

It should be noted that the restriction (4.1) is physically plausible, as it holds true for many cases of shells of revolution, e.g., ellipsoidal, paraboloidal and toroidal.

We now return to (3.14) and observe that condition (3.15) is fulfilled if δ is a constant, and this may be achieved by proper choice of γ and κ.

For shells of variable thickness, as seen from (4.2), condition (3.15) is satisfied provided $(\gamma - \kappa)\ (\mu^2 fh_0/h)^{-1}$ is constant. Hence, by (3.12),

$$\left[\left(\frac{r}{\alpha}\right)h'\right]' - \nu\left[\left(\frac{r'}{\alpha}\right)h\right]' = mK\,\frac{\alpha r}{R_\theta} - \frac{6(1+\nu)}{5}\left(\frac{z'}{R_\theta}\right)h \tag{4.3}$$

which resembles the corresponding equation of the classical theory, first given by Meissner [3] and derived in a different manner recently by Naghdi and DeSilva [9]; in (4.3), K is a constant and the effect of the transverse shear deformation is represented by the second term on the right-hand side.

For shells of uniform thickness, κ vanishes identically and γ in general will not be constant. However, for numerous shell configurations,

$$\delta = 1 - \frac{1}{2}\,i\gamma\left(\mu^2 f\,\frac{h_0}{h}\right)^{-1} \tag{4.4}$$

may be approximated to a constant, in which case the coefficient function Ψ^2 reads

$$\Psi^2 = [1 + i\nu k^{1/2}\lambda]f \tag{4.5}$$

and (4.3) may be reduced to

$$h = \left(\frac{mK}{\nu}\,R_\xi\right)\left[1 - \frac{6(1+\nu)}{5\nu}\,\frac{R_\xi}{R_\theta}\right]^{-1}. \tag{4.6}$$

It follows from (4.6) that γ and δ will in fact be constant if both R_ξ and R_θ are constant. It is clear that this requirement is more restrictive than the corresponding result of the classical theory [9] where $h = (mK/\nu)R_\xi$.

5. Formal solution of Eq. (3.20) by asymptotic integration. We conclude the present paper by discussing formally the solution of the homogeneous differential equation associated with (3.20), namely

$$Y'' + [i^3\mu^2\Psi^2 + \Lambda]Y = \Theta\Phi X_2', \tag{5.1}$$

where μ is a large parameter.

It follows from the work of Langer [8] that corresponding to the homogeneous equation associated with (5.1), that is, the equation

$$Y'' + [i^3\mu^2\Psi^2 + \Lambda]Y = 0, \tag{5.2}$$

there exists a related differential equation whose solution is asymptotic with respect to μ to the solution of (5.2) and that its domain of validity is dependent upon the character of the coefficient functions of Y. If ξ ranges over an interval I_ξ which includes a point ξ_0 at which (i) the function Λ may admit a pole of first or second order*, (ii) Ψ^2 may contain as a factor the quantity $(\xi - \xi_0)^{a-2}$, a being any real positive constant, and (iii) both Λ and Ψ^2 are analytic bounded elsewhere in I_ξ, then the asymptotic solution mentioned will be valid in the entire interval I_ξ including ξ_0. If, on the other hand, the behavior of Λ and Ψ^2 at ξ_0 does not meet the required specifications, then the solution will be valid in a sub-interval of I_ξ.

Thus, if we write Λ and Ψ^2 in the form

$$\Lambda = \frac{A_1}{(\xi - \xi_0)^2} + \frac{B_1}{(\xi - \xi_0)} + \Lambda_1(\xi), \tag{5.3a}$$

$$\Psi^2 = (\xi - \xi_0)^{a-2}\Psi_1^2(\xi), \tag{5.3b}$$

where A_1 and B_1 are constants, Λ_1 is analytic and bounded with respect to μ in I_ξ, and Ψ_1^2 is a non-vanishing single-valued analytic function in I_ξ including ξ_0, then according to Langer [8] the functions

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = (\xi - \xi_0)^{-1/2(a/2-1)}\Psi_1^{-1/2}\varphi^{1/2}\begin{Bmatrix} J_\rho(\varphi) \\ Y_\rho(\varphi) \end{Bmatrix} \tag{5.4}$$

are the solutions of the related differential equation

$$y'' + \left[i^3\mu^2\Psi^2 + \frac{A_1}{(\xi - \xi_0)^2} + \frac{B_1}{(\xi - \xi_0)} + \Omega(\xi)\right]y = 0, \tag{5.5}$$

where Ω is analytic and bounded with respect to μ in I_ξ, J_ρ and Y_ρ are Bessel functions of the first and second kinds, and

$$\rho = \frac{b}{a}, \qquad b = (1 - 4A_1)^{1/2},$$
$$\varphi = i^{3/2}\mu \int_{\xi_0}^{\xi} \Psi(\eta)\, d\eta. \tag{5.6}$$

Writing Λ_1 as

$$\Lambda_1 = \Omega(\xi) + \Delta(\xi), \tag{5.7}$$

where Δ is also analytic and bounded with respect to μ in I_ξ, then (5.2) may be written as

$$Y'' + \left[i^3\mu^2\Psi^2(\xi) + \frac{A_1}{(\xi - \xi_0)^2} + \frac{B_1}{(\xi - \xi_0)} + \Omega(\xi)\right]Y = -\Delta(\xi)Y. \tag{5.8}$$

Since the left-hand side of (5.8) is identical with the left-hand side of the related differential equation (5.5), then, by the method of variation of parameters, there is obtained

$$Y_{H_j} = y_j + \int_{\xi_0}^{\xi} D_1^*(\xi, \eta)[\Delta(\eta)Y_{H_j}(\eta)]\, d\eta; \qquad j = 1, 2, \tag{5.9a}$$

*In cases of ellipsoidal and paraboloidal shells of revolution of uniform thickness, Λ contains a pole of second order at $\xi = 0$.

where

$$D_1^*(\xi, \eta) = \Gamma_1^{-1}(y_1, y_2)[y_1(\xi)y_2(\eta) - y_2(\xi)y_1(\eta)] \tag{5.9b}$$

and Γ_1 denotes the Wronskian of y_1 and y_2.

The integral equation (5.9) by the familiar process of successive iteration leads formally to the relation

$$Y_{H_i} = y_i(\xi) + \sum_{n=1}^{\infty} y_i^{(n)}(\xi), \tag{5.10}$$

where

$$\begin{aligned} y_i^{(n+1)}(\xi) &= \int_{\xi_0}^{\xi} D_1^*(\xi, \eta)\Delta(\eta)y_i^{(n)}(\eta)\, d\eta, \\ y_i^{(0)}(\xi) &= y_i(\xi). \end{aligned} \tag{5.11}$$

The proof for the uniform convergence of $\sum_{n=1}^{\infty} y_i^{(n)}$ in I_ξ is given by Langer [8] and he has further shown that y_i is dominant in (5.10) and thus, y_i is asymptotic with respect to μ to the solution Y_{H_i} of (5.2).

Recalling that δ as given by (4.2) is a complex constant, i.e.,

$$\delta = \delta_1 + i\delta_2 \tag{5.12}$$

then, by (3.19) and (3.18),

$$X_2' = \frac{1}{\delta_1} \mathscr{I}m \left[\frac{Y'}{\Theta} - \frac{\Theta'}{\Theta^2} Y\right], \tag{5.13}$$

where $\mathscr{I}m$ denotes "imaginary part of". Treating (5.1) as a non-homogeneous differential equation whose homogeneous solution is given by (5.10), then again by variation of parameters, there results

$$Y_i = Y_{H_i} - \int_{\xi_0}^{\xi} D_2^*(\xi, \eta)\left\{\frac{1}{\delta_1}\Theta(\eta)\Phi(\eta)\, \mathscr{I}m\left[\frac{Y_i'}{\Theta}(\eta) - \frac{\Theta'}{\Theta^2}(\eta)Y_i(\eta)\right]\right\} d\eta; \qquad j = 1, 2, \tag{5.14a}$$

where

$$D_2^*(\xi, \eta) = \Gamma_2^{-1}(Y_{H_1}, Y_{H_2})[Y_{H_1}(\xi)Y_{H_2}(\eta) - Y_{H_2}(\xi)Y_{H_1}(\eta)] \tag{5.14b}$$

and Γ_2 is the Wronskian of Y_{H_1} and Y_{H_2}.

As in (5.10), by successive iteration, the integral equation (5.14) can be written in the form

$$Y_i = Y_{H_i} - \sum_{n=1}^{\infty} [\chi(\xi)]_i^{(n)}, \tag{5.15}$$

where

$$\begin{aligned} [\chi(\xi)]_i^{(n+1)} &= \int_{\xi_0}^{\xi} D_3(\xi, \eta)\Theta(\eta)\Phi(\eta)[X_2'(\eta)]_i^{(n)}\, d\eta, \\ [X_2'(\xi)]_i^{(0)} &= [X_2'(\xi)]_i, \end{aligned} \tag{5.16}$$

and X_2' is given by (5.13).

The above solution is formally valid if $\sum_{n=1}^{\infty} [\chi]_i^{(n)}$ is uniformly convergent in I_ξ. On account of the interdependence on λ of the various functions involved in (5.16), the proof for the uniform convergence of the series in (5.15) for general shells of revolution appears to be difficult and requires further investigation. Nevertheless, from the comparison of (5.11) and (5.16), it appears reasonable to except that $\sum [\chi]_i^{(n)}$ has in general the same behavior as $\sum y_i^{(n)}$, so that

$$Y_i = Y_{H_i} + O\left(\frac{1}{\mu}\right) \tag{5.17}$$

and hence, y_i is asymptotic with respect to μ to Y_i.

Finally, it may be noted that whenever the right-hand side of (5.1) vanishes identically, as in the case of spherical shells where $\lambda = 0$, the differential equation is of the same form as the corresponding equation of the classical theory (Ref. [9]), although the coefficient functions are not the same.

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THE ACCURACY OF DIFFERENCE APPROXIMATIONS TO PLANE DIRICHLET PROBLEMS WITH PIECEWISE ANALYTIC BOUNDARY VALUES[1]

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1. Introduction. The standard method for appraising the difference between the exact solution u of a problem involving a partial differential equation and the solution U of an approximating finite difference problem is based on the expansion of u by Taylor's formula up to terms of third or higher degree. The partial derivatives of u that enter the argument in this manner are unbounded near the boundary C of the region R where the problem is to be solved, unless C and the prescribed boundary values are very smooth.

In computational practice the boundary and the prescribed boundary values are almost always piecewise analytic. At the corners of C and at the points of C, where the boundary values have a jump in the first or second derivative the higher derivatives of u become usually unbounded. Therefore there prevails the unsatisfactory situation that *most known appraisals of the truncation error $U - u$ in the numerical solution of boundary value problems are based on assumptions which are hardly ever satisfied in computational practice.*

The present note is meant as a first step to overcome this difficulty. It will be shown that for the simplest finite difference approximation to Dirichlet's problem for Laplace's equation the order of magnitude of the truncation error is not affected by jumps in the first derivative of the boundary function. Of course, this is true only outside the immediate neighborhood of those discontinuities.

It is frequently contended that, as the data of a mathematical problem of physical origin are by their very nature only approximate, the discontinuities in the derivatives of boundary values can be safely ignored in problems of this nature. However, there are many problems where these discontinuities are a very accurate model of physical reality, while their elimination by a smooth connecting arc would change either u or U in a manner that cannot be guaranteed to be small without a further investigation like the one given in this paper.

It is hoped that the method of this paper can be extended to more refined finite difference approximations and to other differential problems as well as to problems where the boundary C has corners.

2. Green's function for the difference equation. Let C be a simple closed analytic curve on which is defined a continuous function $f(s)$ that is piecewise analytic. By this statement we mean that C can be described by two analytic functions $x = x(s)$, $y = y(s)$ of period l, regular for real s, and that there exists a finite number of values of s, say $0 \leq s_1 < s_2 \cdots < s_n < l$, such that $f(s)$ is continuous, periodic with period l, and regular analytic in every one of the intervals $s_\nu \leq s \leq s_{\nu+1}$, $\nu = 1, \cdots, n$, and $s_n \leq s \leq s_1 + l$. Furthermore, we require that $(dx/ds)^2 + (dy/ds)^2 \neq 0$.

Denote the interior of C by R. Then there exists a unique function $u(x, y)$ for which

$$\Delta u = 0 \qquad \text{in } R, \qquad u = f \qquad \text{on } C. \tag{1}$$

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The function $u(x, y)$ is regular analytic on C, except possibly at the points S_ν ($\nu = 1, \cdots, n$) of C that correspond to the parameter values s_ν.

We denote by Δ_h the finite difference operator defined by

$$\Delta_h U = h^{-2}[U(x + h, y) + U(x - h, y) + U(x, y + h) + U(x, y - h) - 4U(x, y)].$$

Let R_h be the set consisting of all net points P in R such that the four nearest neighbors of P in the grid lie in $R + C$, and denote by C_h the set of net points in $R + C$ which do not belong to R_h. On C_h we prescribe a function $f_h = f_h(P)$ whose value at a point P of C_h is equal to $f(s)$ at some point P' of C for which $\overline{PP'} < h$.

The problem (1) is to be approximated by

$$\Delta_h U = 0 \qquad \text{in } R_h, \qquad U = f_h \qquad \text{on } C_h. \tag{2}$$

It is known that the truncation error $v = U - u$ is $O(h)$ if $\partial^3 u/\partial x^3$ and $\partial^3 u/\partial y^3$ are uniformly bounded in R. This was first proved by Gerschgorin in [1] by means of the maximum principle for the operator Δ_h, which states that a function U for which $\Delta_h U = 0$ in a set of gridpoints cannot have an extreme value in this set unless it is a constant. Since the third derivatives of u diverge near the points S_ν of C, a stronger tool than the maximum principle is now needed. Our study of the truncation error is based on the asymptotic properties, for $h \to 0$, of *Green's function* for the operator Δ_h in R_h.

For the definition of Green's function we consider the problem

$$\Delta_h V = \varphi(x, y) \qquad \text{in } R_h, \qquad V = 0 \qquad \text{on } C. \tag{3}$$

Its solution is clearly a linear combination of the values of $\varphi(x, y)$ at the points of R_h, so that we may write

$$V(P) = h^2 \sum_{Q \varepsilon R_h} G_h(P, Q)\varphi(Q). \tag{4}$$

Here we have written $\varphi(Q)$ for $\varphi(\xi, \eta)$, etc. The factor h^2 has been extracted in anticipation of a comparison with Green's function for the differential problem. $G_h(P, Q)$ will be called Green's function for the operator Δ_h in R_h. If (3) is interpreted as a system of N linear algebraic equations for the values of $V(P)$ at the N points of R_h, the N^2 values of $h^2 G_h(P, Q)$ form a matrix which is the inverse of the coefficient matrix in the system for $V(P)$. Since the latter matrix is symmetric, so is $G_h(P, Q)$:

$$G_h(P, Q) = G_h(Q, P). \tag{5}$$

By applying (3) and (4) to the particular function $\varphi(Q) = \delta(Q, Q')$, where

$$\delta(Q, Q') = \begin{cases} 0, & Q \neq Q' \\ 1, & Q = Q' \end{cases} \tag{6}$$

it is seen that $G_h(P, Q)$ is the solution of the problem

$$\begin{aligned} \Delta_{h,P} G_h(P, Q) &= h^{-2}\delta(P, Q), \qquad P \varepsilon R_h, \\ G_h(P, Q) &= 0 \qquad\qquad\quad\;\, , \qquad P \varepsilon C_h. \end{aligned} \tag{7}$$

The subscript P in the symbol $\Delta_{h.P}$ means that the operator is to be applied with respect to the variable P. The function $G_h(P, Q)$ is non-negative in R_h. For, $G_h(P, Q)$ cannot

be a constant, because of (7). If it is negative anywhere in R_h, there must be a point $P = P_0$ in R_h where $G_h(P, Q)$, as function of P for fixed Q, has a negative minimum, while $G_h(P_j, Q) > G_h(P_0, Q)$ for at least one of the four nearest neighbors P_j, $(j = 1, \cdots, 4)$, of P_0 in the grid. However, the difference equation in (7) implies that

$$G_h(P_0, Q) \geq \tfrac{1}{4} \sum_{j=1}^{4} G(P_j, Q),$$

and therefore $G_h(P_0, Q)$ must also exceed at least one of the four values $G_h(P_j, Q)$, which is a contradiction to the minimum property of P_0.

The asymptotic study of $G_h(P, Q)$ is based on certain results contained in [2]. It will be convenient to write Sin and Cos instead of the usual symbols sinh, cosh for the hyperbolic function in order to avoid confusion with the trigonometric functions of the mesh length h. We shall be concerned with the function

$$\chi(\sigma, \tau) = \frac{2}{\pi} \int_0^{\pi} \frac{1 - \cos(\tau\lambda)e^{-|\sigma|\mu}}{\sinh \mu} \, d\lambda,$$

where μ is the function of λ defined, for $0 < \lambda < \pi$, by the equation

$$\cos \lambda + \cosh \mu = 2$$

and the condition $\lim \mu/\lambda = 1$. It was shown by McCrea and Whipple [2] that $\chi(\sigma, \tau)$ has the following properties.

1. $\chi(\sigma, \tau) = \chi(\tau, \sigma)$,

2. $\chi(0, 0) = 0$,

3. $\chi(\sigma + 1, \tau) + \chi(\sigma - 1, \tau) + \chi(\sigma, \tau + 1) + \chi(\sigma, \tau - 1) - 4\chi(\sigma, \tau)$
$$= \begin{cases} 0, & \tau^2 + \sigma^2 \neq 0 \\ 4, & \sigma = \tau = 0, \end{cases}$$

4. $\chi(\sigma, \tau) = \dfrac{1}{\pi} \log(\sigma^2 + \tau^2) + c + 0(\sigma^{-2})$, uniformly in τ,

as $\sigma \to \infty$. Here, $c = (1/\pi)(\log 8 + 2\gamma)$, and γ is Euler's constant.

Actually, McCrea and Whipple give only $0(\sigma^{-1})$ as the order of the remainder terms, but their own calculations show that the stronger result is true. For reasons of symmetry we replace the error term by $0\,[1/(\sigma^2 + \tau^2)]$, which is permissible in view of property 1.

It follows, then, that the function

$$\gamma_h(x, y) = \frac{1}{4} \chi\left(\frac{x}{h}, \frac{y}{h}\right) + \frac{1}{2\pi} \log h - c$$

has the properties

1′. $\gamma_h(x, y) = \gamma_h(y, x)$,

2′. $\gamma_h(0, 0) = \dfrac{1}{2\pi} \log h - c$,

$$3'. \quad \Delta_h \gamma_h(x, y) = \begin{cases} 0, & x^2 + y^2 \neq 0 \\ h^{-2}, & x = y = 0, \end{cases}$$

$$4'. \quad \gamma_h(x, y) = \frac{1}{2\pi} \log (x^2 + y^2)^{1/2} + 0\left(\frac{h^2}{x^2 + y^2}\right).$$

The function $-\gamma_h(x, y)$ is a discrete analog of a fundamental solution for Laplace's equation. We now introduce the function

$$H_h(P, Q) = -\gamma_h(x - \xi, y - \eta), \tag{8}$$

where (x, y), (ξ, η) are the coordinates of the points P, Q, respectively. Then we have, by property 3′,

$$\Delta_{h,P} H_h(P, Q) = h^{-2}\delta(P, Q). \tag{9}$$

In order to compare the asymptotic behavior of $H_h(P, Q)$ with that of $G_h(P, Q)$ we introduce the difference

$$e_h(P, Q) = G_h(P, Q) - H_h(P, Q), \tag{10}$$

which, because of (7) and (9), satisfies the difference equation

$$\Delta_{h,P} e_h(P, Q) = 0, \qquad P \in R_h \tag{11}$$

and the boundary condition

$$e_h(P, Q) = -H_h(P, Q), \qquad P \in C_h . \tag{12}$$

We shall show that

$$e_h(P, Q) = \psi(P, Q) + O(h), \qquad \text{for } P \in R_h , \qquad Q \neq P, \qquad \text{and } Q \text{ not on } C, \tag{13}$$

where $\psi(P, Q)$ is, for $P \in R$, the harmonic function of P with boundary values

$$\psi(P, Q) = \frac{1}{2\pi} \log \overline{PQ}, \qquad P \in C. \tag{14}$$

To this end we expand $\Delta_{h,P}\psi(P, Q)$ by Taylor's formula and use the fact that $\psi(P, Q)$ is, for Q not on C, a harmonic function of P in the *closed* domain $R + C$, thanks to the analyticity of the curve C (see [5], p. 187). Then we obtain

$$\Delta_{h,P}\psi(P, Q) = O(h^2), \qquad P \in R_h , \qquad Q \text{ not on } C. \tag{15}$$

Also, by property 4′ and formulas (8) and (14),

$$\psi(P, Q) = -H_h(P, Q) + O(h), \qquad P \in C_h , \qquad P \neq Q. \tag{16}$$

The last two formulas are valid uniformly in Q, if Q is bounded away from C by a positive distance independent of h and if $\overline{PQ} \geq ah^{1/2}$, where a is a positive constant independent of h. If we subtract (15) and (16) from (11) and (12) we find

$$\Delta_{h,P}[e_h(P, Q) - \psi(P, Q)] = O(h^2), \qquad P \in R_h , \qquad Q \neq P, \tag{17}$$

$$e_h(P, Q) - \psi(P, Q) = O(h), \qquad P \in C_h , \qquad P \neq Q. \tag{18}$$

It is well known, and easy to prove by means of the maximum principle for the operator Δ_h , (see [1]) that the Eqs. (17), (18) imply indeed (13).

Finally, we replace $e_h(P, Q)$ by its definition (10), and use once more property 4′. Then we see from (13) that

$$G_h(P, Q) = -\frac{1}{2\pi}\log \overline{PQ} + \psi(P, Q) + O(h), \qquad P \in R_h, \qquad Q \neq P, \qquad Q \text{ not on } C.$$

The first two terms in the right member together constitute Green's function in R for Laplace's operator Δ. Hence,

$$G_h(P, Q) = G(P, Q) + O(h), \qquad \text{for } P \in R_h, \qquad Q \neq P, \qquad Q \text{ not on } C. \tag{19}$$

We repeat that this relation is uniformly valid, if Q is bounded away from C by a positive distance independent of h, and if $\overline{PQ} \geq ah^{1/2}$.

3. The behavior of harmonic functions near corners of the boundary values. We consider first the special case that C is the unit circle and that $u(x, y) = u_m(x, y)$ is a function from the sequence defined recursively by

$$\begin{gathered} u_m(x, y) = Re\, F_m(z), \qquad z = x + iy, \\ F_0(z) = i \log (1 - z), \qquad F_m(z) = -i \int_1^z F_{m-1}(t)\, dt, \qquad t \in R + C. \end{gathered} \tag{20}$$

If

$$1 - z = \rho e^{i\alpha},$$

we take the branch of $\log (1 - z)$ defined by $|\alpha| \leq \pi/2$. Then $F_0(z)$ is regular analytic in $R + C$ except at $z = 1$, and

$$F_0(z) = u_0(x, y) + iv_0(x, y) = -\alpha + i \log \rho.$$

On the boundary,

$$u_0(x, y) = f_0(\theta) = \frac{\pi - \theta}{2}, \qquad 0 < \theta < 2\pi.$$

Hence, $f_0(\theta)$ is continuous except for a jump of size π at $\theta = 0$. For $m > 0$ the functions $u_m(x, y)$ and $v_m(x, y) = \operatorname{Im} F_m(z)$ are continuous in $R + C$. On C we have $z = e^{i\theta}$ and therefore

$$\frac{d^\nu}{d\theta^\nu} F_m(z) = F_{m-\nu}(z)e^{i\nu\theta} + \cdots, \qquad z \in C,$$

i.e. the boundary values $f_m(\theta)$ of $u_m(x, y)$ have the derivatives

$$f_m^{(\nu)}(\theta) = f_{m-\nu}(\theta) \cos \nu\theta - v_{m-\nu}(x, y) \sin \nu\theta + \cdots, \qquad (x, y) \in C,$$

where the dots indicate terms involving higher subscripts. This proves the continuity of $f_m^{(\nu)}(\theta)$ for $\nu < m$. For $\nu = m$, the first righthand term has a jump of size $(-1)^m\pi$ at $\theta = 0$. The second term equals $\log \rho \sin \nu\theta$, which has the limit 0 at $\theta = 0$. All the other terms are continuous, so that the jump of $(-1)^m\pi$ characterizes the behavior of $f_m^{(m)}(\theta)$ at $\theta = 0$.

The partial derivatives of $u_m(x, y)$ of order $\nu < m$ are continuous in $R + C$. This follows directly from the definition of $u_m(x, y)$ in (20) and the Cauchy-Riemann equations. The mth derivatives are multiples of $u_0(x, y)$ or $v_0(x, y)$ and are therefore $O(\log \rho)$,

at worst. If $\nu > m$, the derivatives are—except for numerical factors—the real or imaginary part of $d^{\nu-m}/dz^{\nu-m} \log (1 - z)$. Their order of magnitude is, thus, $0(\rho^{\nu-m})$.

Now assume that the boundary function $f(s) = f(\theta)$ of $u(x, y)$ is regular analytic on the unit circle C except at $\theta = 0$, (i.e. $z = 1$), where it has a jump of size K in the pth derivative, while the derivatives of lower order are continuous. The function $f(\theta) - (-1)^p(K/\pi)f_p(\theta)$ has then continuous derivatives on C up to order p inclusive. If it has a discontinuity in the $(p + 1)$st derivative, this can also be eliminated by addition of a suitable multiple of $f_{p+1}(\theta)$, and so forth. Hence, there exists a linear combination

$$f^*(\theta) = f(\theta) + \alpha_0 f_p(\theta) + \alpha_1 f_{p+1}(\theta) + \cdots + \alpha_q f_{p+q}(\theta)$$

with continuous derivatives of order $p + q$ and, at worst, a jump in the $(p + q + 1)$st derivative. The harmonic function with boundary values $f^*(\theta)$ on C possesses then continuous derivatives up to order $p + q$, inclusive in $R + C$ (see [3], p. 243]. Therefore

$$\begin{aligned}\frac{\partial^{p+q}u}{\partial x^{p+q}} &= O\left[\alpha_0 \frac{\partial u_p^{p+q}}{\partial x^{p+q}} + \cdots + \alpha_q \frac{\partial^{p+q}u_{p+q}}{\partial x^{p+q}}\right] \\ &= O\left[\frac{\partial u_p^{p+q}}{\partial x^{p+q}}\right] = \begin{cases} O(\log \rho), & q = 0 \\ O(\rho^{-q}) \ , & q > 0 \end{cases}\end{aligned} \tag{21}$$

and, similarly,

$$\frac{\partial^{p+q}u}{\partial y\, \partial x^{p+q-1}} = \begin{cases} O(\log \rho), & q = 0, \\ O(\rho^{-q}) \ , & q > 0. \end{cases} \tag{22}$$

A similar discontinuity of a derivative of $f(\theta)$ at any other point $\theta = \theta_0$ on the unit circle C can be analyzed by means of auxiliary harmonic functions obtained from $u_m(x, y)$ by a rotation through the angle θ_0 . If more than one point with discontinuities in the derivatives of $f(\theta)$ occur they can be handled simultaneously by adding a sum of appropriate compensating harmonic functions. Finally, if C is not the unit circle R can be changed into the interior of the unit circle by a conformal mapping. The mapping function and its inverse transform harmonic functions in R into harmonic functions inside the unit circle, and conversely. Since C is analytic the mapping function is analytic in $C + R$ and hecne the transformed boundary function is harmonic at all points of the unit circle except those corresponding to the points S_j , $j = 1, \cdots , n$ on C. Furthermore, the orders of magnitude of the harmonic function in R near the points S_j are the same as those of the image function in the unit circle. Hence, we have proved that *the solution $u(x, y)$ of problem* (1) *is harmonic in $R + C$ except at the boundary points S_j , where $f(\theta)$ is not analytic. At those points the order of magnitude of the derivatives for approach in $R + C$ is determined by the formulas* (21) *and* (22).

4. The truncation error for Dirichlet's problem. The truncation error $v = U - u$ for the approximate solution of problem (1) by means of problem (2) is the solution of the problem

$$\Delta_h v = -\Delta_h u \ , \qquad \text{in } R_h \ , \tag{23}$$

$$v = f_h - u, \qquad \text{on } C_h \ . \tag{24}$$

By Taylor's formula we have

$$\Delta_h u(x, y) = \frac{h}{6}\left[\frac{\partial^3}{\partial x^3} u(x + \theta h, y) - \frac{\partial^3}{\partial x^3} u(x - \theta h, y) + \frac{\partial^3}{\partial y^3} u(x, y + \theta h) - \frac{\partial^3}{\partial y^3} u(x, y - \theta h\right] \tag{25}$$

with $0 \leq \theta < 1$. We know from the preceding section that $\partial^3/\partial x^3\, u(Q)$ and $\partial^3/\partial y^3\, u(Q)$ are $0(\overline{QS_i}^{-2})$ near S_i. For simplicity we consider only the case that there is no more than one singular point $S_i = S$ on C, and set $\overline{QS} = \rho$. The extension to a finite number of such points is trivial.

From now on we must subject the choice of grids to the important restriction that *the distance of the points S_i on C from the nearest grid line be at least bh, $0 < b < 1$, where b is independent of h.* Then the right member of (25) does not exceed $K_1 h\rho^{-2}$ in R, where K_1 is a constant depending on $f(s)$ and on b, not on h, i.e.

$$| \Delta_h u | \leq K_1 h \rho^{-2}. \tag{26}$$

In order to arrive at an appraisal of v we represent v as the sum $v = v_1 + v_2$ of the solutions of the problems

$$\begin{aligned} \Delta_h v_1 &= -\Delta_h u, && \text{in } R_h\,, \\ v_1 &= 0 \quad , && \text{on } C_h \end{aligned} \tag{27}$$

and

$$\begin{aligned} \Delta_h v_2 &= 0 \quad , && \text{in } R_h\,, \\ v_2 &= f_h - u, && \text{on } C_h\,. \end{aligned} \tag{28}$$

Discussing v_1 first we represent this function in the form (4) and make use of (26), obtaining

$$| v_1(P) | \leq hK_1 \sum_{Q \in R_h} h^2 G_h(P, Q)\rho^{-2}. \tag{29}$$

If we extend the definition of $G_h(P, Q)$ in an appropriate manner from the points of $R_h + C_h$ to all points of $R + C$, the sum in the right member can be replaced by an integral. To this end we associate with every square of the grid that lies entirely in $R + C$ the value of $G_h(P, Q)$ as function of Q, at its lower left vertex. At the remaining points of R we define $G_h(P, Q)$ as being zero. Then

$$\sum_{Q \in R_h} h^2 G_h(P, Q)\rho^{-2} = \iint_R G_h(P, Q)\rho^{-2}\, dQ + O(h). \tag{30}$$

Since $G_h(P, Q)$ is symmetric in P and Q, we can appraise the integral above by means of the asymptotic formula (19), *provided P is restricted to some proper closed subdomain R' of R.* Let

$$R = R_1 + R_2 + R_3\,,$$

where R_1 is the circular region $\overline{PQ} < ah^{1/2}$ about P, the domain R_2 is closed and satisfies $R' \subset R_1 + R_2 \subset R$, and, finally, $R_3 = R - R_1 - R_2$.

We show first that

$$\iint_{R_1} G_h(P, Q)\rho^{-2}\, dQ = O(h \log h). \tag{31}$$

To see this we observe that the function $| e_h(P, Q) |$ assumes its maximum on C_h, because of (11). In view of (8), (12) and the definition of $\gamma_h(x, y)$ it follows that $| e_h(P, Q) |$ remains uniformly bounded, for $Q \varepsilon R'$, as $h \to 0$. Since $H_h(P, Q) = O(\log h)$, in consequence of property 2′, we see from (10) that

$$G_h(P, Q) = O(\log h), \tag{32}$$

if $Q \varepsilon R'$ or if $P \varepsilon R'$ (the latter because of the symmetry of Green's function). This proves (31).

Furthermore

$$\iint_{R_2} G_h(P, Q)\rho^{-2}\, dQ = \iint_{R_2} G(P, Q)\rho^{-2}\, dQ + \iint_{R_2} O(h)\rho^{-2}\, dQ, \tag{33}$$

by (19). The first integral in the right member exists, because ρ^{-1} is bounded in R_2, and the last integral is $O(h)$. Thus

$$\iint_{R_2} G_h(P, Q)\rho^{-2}\, dQ = O(1). \tag{34}$$

In the remaining integral $\iint_{R_3} G_h(P, Q)\rho^{-2}\, dQ$ we make use of the fact that $G_h(P, Q)$ is, by definition, zero whenever Q is a point of a gridsquare that does not lie entirely in $R + C$. The point S lies in such a square; in fact, it has at least the distance bh from the edges of this square by virtue of the hypothesis introduced in the paragraph after formula (25). Therefore, R_3 may be replaced in the integration by a subregion R_3^* that has at least the distance bh from S. Using again (19), it follows that

$$\iint_{R_3} G_h(P, Q)\rho^{-2}\, dQ = \iint_{R_3^*} G(P, Q)\rho^{-2}\, dQ + \iint_{R_3^*} O(h)\rho^{-2}\, dQ.$$

The first integral in the right member is bounded, because $G(P, Q)$ vanishes on the boundary and is there continuously differentiable so that

$$G(P, Q) < K_2\rho, \qquad P \varepsilon R', \qquad Q \varepsilon R_3 .$$

The last integral in the right member does not exceed

$$b^{-1} \iint_{R_3^*} | O(h) |\, h^{-1}\rho^{-1}\, dQ,$$

which is bounded. Therefore

$$\iint_{R_3} G_h(P, Q)\rho^{-2}\, dQ = O(1). \tag{35}$$

If (31), (34) and (35) are inserted into (30), the inequality (29) is seen to imply that

$$v_1(P) = O(h), \qquad P \varepsilon R'. \tag{36}$$

We now turn to the appraisal of v_2. In order to make our analysis of Green's function available for this problem, we use Green's formula for the operator Δ_h, which can be written

$$h^2 \sum_{R_h} (V\Delta_h U - U\Delta_h V) + h \sum_{C_h} (V\Gamma_h U - U\Gamma_h V) = 0, \tag{37}$$

(see [4], p. 36). Here U and V may be any two functions in the grid, and the operator $\Gamma_h U$ is defined as follows: Let Q be a point of C_h and denote by Q_j, ($j = 1, \cdots \nu \leq 3$) the gridpoints in R_h at distance h from Q, then

$$\Gamma_h U(Q) = h^{-1}[U(Q_1) + \cdots + U(Q_\nu) - \nu U(Q)]. \tag{38}$$

We apply (37) with $V = v_2(Q)$, $U = G_h(P, Q)$ and find, using (7) and (28), that

$$v_2(P) = -h \sum_{Q \epsilon C_h} \Gamma_{h,Q}[G_h(P, Q)][f_h(Q) - u(Q)]. \tag{39}$$

Now, by definition, $f_h(Q) = f(Q')$ where Q' lies on C, and $\overline{QQ'} < h$. In analogy with the hypothesis introduced after formula (25), *we have to restrict the choice of Q' somewhat by requiring that the distance from S to the points of the segment QQ' be at least bh uniformly for Q on C_h. By the theorem of the mean*

$$f_h(Q) - u(Q) = f(Q') - u(Q) = \frac{\partial}{\partial\sigma} u(Q'') \cdot \overline{QQ'},$$

where $\partial/\partial\sigma$ indicates differentiation in the direction from Q' to Q and Q'' is a point between Q and Q'. We proved in Sec. 3 that the first derivative of u at the point Q'' in any direction is $0(\log \rho'')$, the letter ρ'' designating the distance $\overline{SQ''}$. With the help of our restriction on Q' we shall show first that

$$| f_h(Q) - u(Q) | \leq K_3 h(| \log \rho | + 1), \tag{40}$$

K_3 being independent of h and ρ. By assumption, $\rho \geq bh$ and $\overline{QQ''} < \overline{QQ'} \leq h$, whence $\overline{QQ''} \leq \rho/b$. Therefore, $\rho'' \leq \rho + \overline{QQ''} \leq \rho(1 + 1/b)$. Using our other assumption that $\rho'' \geq bh$ we show analogously that $\rho \leq \rho'' + QQ'' < \rho''(1 + 1/b)$. The ensuing double inequality

$$\log \rho - \log\left(1 + \frac{1}{b}\right) \leq \log \rho'' \leq \log \rho + \log\left(1 + \frac{1}{b}\right)$$

implies that

$$| \log \rho'' | \leq \max\left\{\left| \log \rho + \log\left(1 + \frac{1}{b}\right) \right|, \left| \log \rho - \log\left(1 + \frac{1}{b}\right) \right|\right\}$$
$$\leq | \log \rho | + \log\left(1 + \frac{1}{b}\right),$$

and, therefore,

$$\left| \frac{\partial}{\partial\sigma} u(Q'') \cdot \overline{QQ'} \right| = O(h \log \rho'') \leq K_3 h(| \log \rho | + 1).$$

On C_h the relation (19) can be used, if $P \epsilon R'$, and this implies that $G_h(P, Q)$ is $O(h)$, for $Q \epsilon C_h$. Using (38) and (40) we obtain then from (39) the inequality

$$| v_2(P) | \leq K_4 h \sum_{C_h} h(| \log \rho | + 1), \qquad (K_4 \text{ a constant}). \tag{41}$$

It is plausible that

$$\sum_{C_h} h(| \log \rho | + 1) = O(1). \tag{42}$$

The simple proof of this statement will be postponed to the end of this section. On the basis of (41) and (42) we have

$$| v_2(P) | = O(h), \qquad P \varepsilon R'.$$

This relation, together with (36) completes the proof of our main result, which we now state as a formal theorem.

Theorem: The truncation error $v(P) = U(P) - u(P)$ *corresponding to the approximate solution of problem* (1) *by means of the equations* (2) *is of order* $O(h)$, *provided the following conditions are satisfied*:

(*a*) *the boundary* C *is a simple closed analytic curve;*
(*b*) *the boundary function* $f(s)$ *is continuous and piecewise analytic;*
(*c*) *the distance from the singularities of* $f(s)$ *to the nearest point on a grid line is not less than* bh, $(0 < b < 1$, *independent of* $h)$.
(*d*) *If* $f_h(Q) = f(Q')$, $Q' \varepsilon C$, *then the distance of the singularities of* $f(s)$ *from the segment* QQ' *is not less than* bh.

The truncation error $v(P)$ *has the order* $O(h)$ *uniformly in every closed subdomain of* R.

Proof of formula (42). We show first that the total number M of the gridpoints in C_h is $O(h^{-1})$. The analytic curve C possesses only a finite number of points where either dx/ds or dy/ds vanishes. These points divide C into a finite number of arcs each of which does not intersect the same grid line twice. Hence, if C^0 is one such arc and if the lengths of its projections on the axes are L_x and L_y, respectively, then the arc C^0 possesses at most $(L_x + L_y + 2)h^{-1}$ points of intersection with lines of the grid. It follows that the total number of intersections of C with lines of the grid is $O(h^{-1})$. Now, every point of C_h is an end point of a mesh side of length h that has a point in common with C. Hence, $M = O(h^{-1})$, as claimed, say,

$$M \leq Lh^{-1}.$$

For sufficiently small h no closed segment of length h joining two gridpoints will have more than one point in common with C. Then exactly one of the two end points of such a segment belongs to C_h. We now measure the arc length s on C from S and order the points P_r of C_h in such a way that $r_1 < r_2$ if and only if P_{r_1} is an end point of a segment that meets C at a point with smaller value of s than any grid segment ending at P_{r_2}. Then $\overline{SP_r} \leq 2^{1/2} rh$ and, therefore,

$$\sum_{C_h} h(| \log \rho | + 1) \leq \sum_{r=1}^{M} h(| \log 2^{1/2} rh | + 1)$$

$$\leq \int_0^{L+h} (| \log 2^{1/2} t | + 1)\, dt \leq K_5, \qquad (K_5 \text{ a constant}),$$

which proves (42).

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BOOK REVIEWS

(*Continued from p. 40*)

Boundary layer effects in aerodynamics. Proceedings of a Symposium held at the National Physical Laboratory on 31st March and 1st April, 1955. Her Majesty's Stationary Office, London, 1955. ix + 400 pp. $4.00.

This book is a collection of nine papers presented at an international symposium on boundary layers held at the National Physical Laboratory in 1955. Included in the compilation is an introductory address by L. Howarth and a summary of the discussion which followed each presentation. The papers covered most of the topics currently under investigation in the field, and their character varied from those of an applied mathematical and fundamental physical nature, to those concerned more than with the practical engineering aspects of boundary layer studies. This fine collection would seem to make a basic contribution at each of these levels. An interesting feature of the compilation proves to be the discussions which appear in summarized form. Unfortunately, some of the utility of the book is lost by the fact that nearly all the papers have recently appeared in the journals, or in other collected aerodynamic studies.

The following is a list of the authors and the general topic of the paper: R. Timman (three-dimensional boundary layers), M. B. Glauert and M. J. Lighthill (axisymmetric boundary layers), M. Gregory, J. T. Stuart and W. S. Walker (stability of three-dimensional boundary layers), G. B. Schubauer and P. S. Klebanoff (transition), D. Küchemann (viscosity effects on swept wings), R. C. Pankhurst (boundary layer control), A. D. Young and S. Kirby (profile drag at supersonic speeds), D. W. Holder and G. E. Gadd (shock-boundary layer interaction), H. H. Pearcey (transonic turbulent separation on airfoils).

Ronald F. Probstein

Thermodynamics and statistical mechanics. By Arnold Sommerfeld. Academic Press, Inc., New York, 1956. xviii + 401 pp. $7.00.

For anyone who is familiar with the Lectures on Theoretical Physics—Vols. I, II, III, IV, VI, by A. Sommerfeld—it is a sufficient review of the present book merely to say that it is Vol. V of the series. The remarkable understanding of theoretical physics by the author is in this volume as in the others made available to the reader by the clear and simple way in which the ideas are ordered and presented.

As the name implies the book presents both the macroscopic thermodynamics and the classical and quantum statistical mechanics. About the first 2/5ths of this 400 page book is on thermodynamics. The last 3/5ths is on statistical mechanics. At the end of the first portion is a brief presentation (16 pages) of "Irreversible Processes: Thermodynamics of near-equilibrium processes".

Any book of 400 pages on a subject with an extensive literature must be selective in its material. The reviewer regards the selection of material for this volume as good. The first chapter on "Thermodynamics—General Considerations" is followed by Chapter II on "The Application of Thermodynamics to Special Systems". In the closing section of this chapter irreversible processes are treated. "The Elementary Kinetic Theory of Gases" is treated in Chapter III. Next follows a long Chapter IV on "General Statistical Mechanics: Combinatorial Method". Here the basic ideas and their applications to a number of special problems is treated in most lucid detail. The last, Chapter V, gives an "Outline of an Exact Kinetic Theory of Gases", where applications are carried through Conductivity and the Wiederman-Franz Law where the author's own basic contributions are presented in proper perspective.

As many of the readers of this review will know, the author died before he could finish this book. Several sections were completed by Bopp and Meixner who are to be congratulated on their ability to finish a work by a master book writer and great theoretical physicist without discontinuities showing. Although the reviewer has not attempted a comparison of the original German and translated English versions, the latter reads exceptionally well and gives every indication of being a faithful as well as fluent translation.

H. Emmons

(*Continued on p. 82*)

BIHARMONIC EIGENVALUE PROBLEM OF THE SEMI-INFINITE STRIP*

BY

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Abstract. A basic problem in the evaluation of residual stresses in simple elastic structures concerns the determination of the stress and deformation state produced by self-equilibrating, but otherwise arbitrary, normal and shear tractions acting on the edge $x = 0$ of a semi-infinite elastic strip $(0 \leq x \leq \infty, -1 \leq y \leq 1)$ which is free along the edges $y = \pm 1$. This strip is known to experience, in accordance with St. Venant's principle, inappreciable stresses at distances $x \gtrsim 2$ from the loaded edge, in spite of the very large stresses it may experience in the vicinity of the edge. An earlier paper [*The end problem of rectangular strips*, J. Appl. Mech. (1953)] based on the variational principle, established approximate eigenfunctions (modes of response) and eigenvalues (laws of oscillation and decay) for the various possible self-equilibrating end tractions. In this paper we give a rigorous solution of the end problem. This solution is obtained in two steps. First we solve the two "mixed" end problems: the parallel edges $y = \pm 1$ of the strip are free, and along the vertical edge $x = 0$ (a) the shear displacement is given, the normal stress is zero, (b) the normal displacement is given, the shear stress is zero. These two problems are solved by extending the strip to the left, to $-\infty$, and finding the tractions that must be applied at $y = \pm 1$ $(x < 0)$ and at $x = -\infty$, so that one have $\sigma_x = 0$, $\tau = 0$, respectively, at $x = 0$, while the edge values of the displacements (more specifically, of dv/dy and u) are orthogonal polynomials in y (Horvay-Spiess polynomials and Legendre polynomials, respectively). The corresponding stress functions $K_n(x, y)$, $J_n(x, y)$ are found in the form of Fourier integrals plus polynomial terms. For $x \geq 0$ they may be rewritten as real parts of $\sum C_{Knk} \Phi_k$, $\sum C_{Jnk} \Phi_k$, where $\Phi_k = z_k^{-2} e^{-z_k x} (\cos z_k y - y \cot z_k \sin z_k y)$ or $z_k^{-2} e^{-z_k x} (\sin z_k y - y \tan z_k \cos z_k y)$, and $\sin 2z_k \pm 2z_k = 0$. An alternate procedure for determining the coefficients C_{Knk}, C_{Jnk}, based on a formula of R. C. T. Smith, which bypasses the extension of the strip to $x = -\infty$, is also furnished. The second phase of the solution of the "pure" end problem—along the short edge (a) the shear stress is given, normal stress is zero, (b) the normal stress is given, shear stress is zero—consists in recombining the biharmonic eigenfunctions K_n, J_n within each class into functions $H_n(x, y)$, $G_n(x, y)$, so that the $x = 0$ values of $H_{n,xy}$, $G_{n,yy}$ constitute two complete orthonormal sets of (transcendental) functions in y into which the given boundary stresses may be expanded.

1. Introduction. We propose to solve the biharmonic eigenvalue problem of the semi-infinite strip. More specifically, we shall establish functions $H_n(x, y)$, $G_n(x, y)$ (even in y for even n, odd in y for odd n) such that

(a) H_n, G_n are biharmonic functions

$$\nabla^4 H_n = \nabla^4 G_n = 0. \tag{1}$$

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(b) H_n, G_n satisfy homogeneous boundary conditions of zero normal stress, zero shear stress along the long edges (star denotes boundary value at $y = +1$)[1]:

$$H^*_{n,xx} = G^*_{n,xx} = 0, \qquad H^*_{n,xy} = G^*_{n,xy} = 0. \tag{2}$$

(c) H_n gives zero normal stress, G_n gives zero shear stress along the short edge (the "degree" sign denotes boundary value at $x = 0$)

$$H^\circ_{n,yy} = 0, \qquad G^\circ_{n,xy} = 0. \tag{3}$$

(d) The edge values

$$t_n(y) \equiv -H^\circ_{n,xy}, \qquad s_n(y) \equiv G^\circ_{n,yy} \tag{4}$$

constitute two complete orthonormal sets of functions into which prescribed self-equilibrating edge tractions τ°, σ°_x, i.e., tractions satisfying the conditions

$$\int_{-1}^{+1} \tau^\circ \, dy = \int_{-1}^{+1} \sigma^\circ_x \, dy = \int_{-1}^{+1} \sigma^\circ_x y \, dy = 0 \tag{5}$$

may be expanded:

$$\tau^\circ(y) = \sum_n \langle \tau^\circ, t_n \rangle t_n(y), \qquad \sigma^\circ_x(y) = \sum_n \langle \sigma^\circ_x, s_n \rangle s_n(y). \tag{6}$$

It follows that

$$\begin{aligned} H(x, y) &= \sum_n \langle \tau^\circ, t_n \rangle H_n(x, y) \\ G(x, y) &= \sum_n \langle \sigma^\circ_x, s_n \rangle G_n(x, y) \end{aligned} \tag{7a,b}$$

are the stress functions of the two problems. We used the notation

$$\langle f, g \rangle \equiv \int_{-1}^{+1} f(y) g(y) \, dy, \qquad | f | \equiv \langle f, f \rangle^{1/2}. \tag{8}$$

We arrive at the solutions H_n, G_n of the "pure" end problems by first solving the easier, "mixed" end problems pertaining to determination of stress functions K_n and J_n, which comply with conditions (a), (b), and with the modified conditions (c′) and (d′)[2]:

(c′) $$K^\circ_{n,yy} = 0, \qquad I^\circ_{n,xx} = 0, \qquad (I_{n,x} \equiv J_n), \tag{9}$$

(d′) $$K^\circ_{n,xx} = 1 - y^2,\ y - y^3,\ 1 - 8y^2 + 7y^4,\ \cdots \qquad (n = 2, 3, 4, \cdots) \tag{10}$$

$$I^\circ_{n,yy} = (-1 + 3y^2)/2,\ (-3y + 5y^3)/2,\ \cdots . \tag{11}$$

The functions $K^\circ_{n,xx}$ will be recognized as the (unnormalized) Horvay-Spiess polynomials Q_2, Q_3, Q_4, $\cdots$ (denoted formerly, except for normalization factors, by f_0,

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[1]Clearly, because of the evenness or oddness of the functions H_n, G_n, the data at $y = +1$ also specify the data at $y = -1$. We use as distance unit the semiwidth of the strip, as stress unit the modulus of elasticity.

[2]The advantage of regarding I_n rather than $J_n \equiv I_{n,x}$ as the basic function will become apparent in the sequel.

f_1, f_2, $\cdots$; see [1], Eq. (12)), with leading coefficient chosen as 1, which constitute a complete orthogonal set with respect to the boundary condition

$$Q_n(+1) = 0 \tag{12}$$

while the functions $I^\circ_{n.yy}$ are the well-known Legendre polynomials P_2, P_3, P_4, $\cdots$ forming a complete orthogonal set with respect to the boundary condition

$$P_n(+1) = 1. \tag{13}$$

We disregard the first two Legendre polynomials, $P_0 = 1$, $P_1 = y$, which represent rigid body motions, and do not give rise to stresses. In contrast, the singular functions

$$Q_0 = 1, \qquad Q_1 = y \tag{14}$$

which violate condition (12) (the condition of $\sigma_x(0, \pm 1) = \sigma_y(0, \pm 1) = 0$) are of considerable interest. It should be remembered that the corresponding stress functions, K_0, K_1 may be written as linear combinations $\sum_2^\infty c_i K_i$ of the complete set of functions K_n, $n \geq 2$, and so their separate consideration is somewhat redundant. Nevertheless, a direct determination of K_0, K_1 is of great practical value; we shall therefore list their direct formulas along with those of K_2, K_3, $\cdots$. (The symbol K_0 was previously denoted by 2Φ in [4] and by $2\mathcal{H}_1$ in [7], the symbol J_2 was previously denoted by $6\mathcal{G}$ in [7].)

It is clear that solution of the (a), (b), (c′), (d′) problem resolves the end problem of given shear displacement, zero normal stress, and given normal displacement, zero shear stress. For, if K is a stress function, then

$$K_{.yy} = \sigma_x , \qquad K_{.xx} = \nu K_{.yy} + dv/dy \tag{15}$$

and, because of condition (c′), the edge values

$$\begin{bmatrix} K^\circ_{.xx} \\ K^\circ_{.yy} \end{bmatrix} = \begin{bmatrix} dv^\circ/dy \\ 0 \end{bmatrix} \tag{16}$$

are properly specified. Similarly for a stress function $I_{.x}$ we have

$$I_{.xx} = -\int_{-1}^{y} \tau \, dy, \qquad I_{.yy} = \nu I_{.xx} + u, \tag{17}$$

hence, because of condition (c′), the edge values

$$\begin{bmatrix} I^\circ_{.xx} \\ I^\circ_{.yy} \end{bmatrix} = \begin{bmatrix} 0 \\ u^\circ \end{bmatrix} \tag{18}$$

are again properly specified. Thus, the two mixed end problems have the stress functions

$$\begin{aligned} K(x, y) &= \sum_2^\infty \frac{\langle K^\circ_{n.xx} , dv^\circ/dy \rangle}{| K^\circ_{n.xx} |^2} K_n(x, y) \\ J(x, y) \equiv I_{.x}(x, y) &= \sum_2^\infty \frac{\langle I^\circ_{n.yy} , u^\circ \rangle}{| I^\circ_{n.yy} |^2} I_{n.x}(x, y) \end{aligned} \tag{19}$$

as solutions[3].

[3]It is obvious that the edge value problem—I and $\nabla^2 I$ are specified along $x = 0$— is also solved in terms of Eqs. (19).

Once we have determined the functions $K_n(x, y)$, $J_n(x, y)$ appropriate to the mixed end problems, it is easy to calculate the shear stress boundary values $-K^{\circ}_{n.xy}$ and the normal stress boundary values $J^{\circ}_{n.yy}$. We shall obtain these in integral form

$$-\tau^{\circ}_n \equiv K^{\circ}_{n.xy} = \frac{2}{\pi}\int_0^{\infty}\left[\mathcal{A}(\lambda)\begin{Bmatrix} ch\ \lambda y \\ sh\ \lambda y \end{Bmatrix} + \mathcal{B}(\lambda)y\begin{Bmatrix} sh\ \lambda y \\ ch\ \lambda y \end{Bmatrix} + \mathcal{P}(\lambda, y)\right]\frac{d\lambda}{\lambda^n}, \tag{20}$$

$\sigma^{\circ}_n \equiv J^{\circ}_{n.yy} =$ similar,

where $\mathcal{P}(\lambda, y)$ represents polynomial terms in λ and y, progressing to such power in λ as to compensate for the singularities introduced by the terms $\mathcal{A}/\lambda^n$, $\mathcal{B}/\lambda^n$ (see Table II). Knowing the integral representations of $\tau^{\circ}_n(y)$ it is easy to orthonormalize these expressions, writing

$$\begin{aligned} t_2(y) &= b_{22}\tau^{\circ}_2(y), \\ t_4(y) &= b_{44}\tau^{\circ}_4(y) + b_{42}\tau^{\circ}_2(y), \\ t_6(y) &= b_{66}\tau^{\circ}_6(y) + b_{64}\tau^{\circ}_4(y) + b_{62}\tau^{\circ}_2(y) \\ &\cdot \qquad \cdot \qquad \cdot \end{aligned} \tag{21}$$

and choosing the b_{nk} so that (δ_{nm} = Kronecker delta)

$$\langle t_n\,,\, t_m\rangle = \delta_{nm}\,. \tag{22}$$

It follows that

$$\begin{aligned} H_2(x, y) &= b_{22}K_2(x, y), \\ H_4(x, y) &= b_{44}K_4(x, y) + b_{42}K_2(x, y) \\ &\cdot \qquad \cdot \qquad \cdot \end{aligned} \tag{23}$$

are the stress functions which resolve the edge value problem τ° = given, $\sigma^{\circ}_x = 0$, in the form of (7a), and

$$\begin{aligned} G_2(x, y) &= a_{22}J_2(x, y), \\ G_4(x, y) &= a_{44}J_4(x, y) + a_{42}J_2(x, y), \\ &\cdot \qquad \cdot \qquad \cdot \end{aligned} \tag{24}$$

where the a_{nk} are determined from the requirement that

$$\begin{aligned} s_2(y) &= a_{22}\sigma^{\circ}_2(y), \\ s_4(y) &= a_{44}\sigma^{\circ}_4(y) + a_{42}\sigma^{\circ}_2(y), \\ &\cdot \qquad \cdot \qquad \cdot \\ \langle s_n\,,\, s_m\rangle &= \delta_{nm} \end{aligned} \tag{25}$$

are the stress functions which resolve the edge value problem σ°_x = given, $\tau^{\circ} = 0$, in the form of (7b).

2. The functions K_n, I_n, H_n, G_n. Let z_k denote the first quadrant roots $z_2, z_4, \cdots$, and $z_3, z_5, \cdots$ of

$$\sin 2z_k + 2z_k = 0, \qquad \sin 2z_k - 2z_k = 0 \tag{26}$$

respectively. We recall that the Fadle-Papkovitch solutions [5], [6], [2], of $\nabla^4\Phi = 0$ $(x \geq 0)$,

$$\begin{aligned} \Phi_k(x, y) &= z_k^{-2}e^{-z_k x}(\cos z_k y - y \cot z_k \sin z_k y) \qquad (k = 2, 4, 6, \cdots) \\ \Phi_k(x, y) &= z_k^{-2}e^{-z_k x}(\sin z_k y - y \tan z_k \cos z_k y) \qquad (k = 3, 5, 7, \cdots) \end{aligned} \tag{27}$$

satisfy the homogeneous stress conditions (2) along the long edges of the strip, but produce *both* σ_x° and τ° values, and *both* u° and v° displacements along the short edge. What is desired are, however, such combinations of Φ_k which give *either* σ_x° stress *or* τ° stress, making the other stress zero[4]. Such are the functions K_n, J_n and H_n, G_n.

In the remainder of this section we give the results of the analysis. The analysis itself will be carried out in Secs. 3 and 4.

In Table I below we list, up to $n = 6$, the expansion coefficients in terms of the Fadle-Papkovitch functions Φ_k, of the eigenfunctions

$$K_n(x, y) = \Re \sum_k C_{nk}\Phi_k(x, y) \tag{28}$$

TABLE I. Formulas for the coefficients C_{nk} in the expansions of K_n and I_n, Eqs. (28), (29). For n = even, $C'_{nk} \equiv C_{nk} \cos^3 z_k/\sin^2 z_k$, for n = odd, $C'_{nk} \equiv C_{nk}/\cos z_k$ is shown. The subscript k of z_k is omitted for the sake of simpler notation.

K_n	dv°/dy	C'_{nk}
K_0	1	2
K_1	y	$-2z^{-1}$
K_2	$1 - y^2$	$4z^{-2}(2 + \cos^2 z)$
K_3	$y - y^3$	$-4z^{-3}(6 - \cos^2 z)$
K_4	$1 - 8y^2 + 7y^4$	$8z^{-4}[42(3 + 2\cos^2 z) - z^2(34 + 3\cos^2 z)]$
K_5	$y - 4y^3 + 3y^5$	$-8z^{-5}[30(9 - 2\cos^2 z) - z^2(18 - \cos^2 z)]$
K_6	$1 - 19y^2 + 51y^4 - 33y^6$	$32z^{-6}[1485(4 + 3\cos^2 z)$ $- 18z^2(111 + 19\cos^2 z) + 2z^4(13 + \cos^2 z)]$

I_n	u°	C'_{nk}
I_2	$(-1 + 3y^2)/2$	$-6z^{-2}$
I_3	$(-3y + 5y^3)/2$	$30z^{-3}$
I_4	$(3 - 30y^2 + 35y^4)/8$	$30z^{-4}[7(2 + \cos^2 z) - 3z^2]$
I_5	$(15y - 70y^3 + 63y^5)/8$	$-210z^{-5}[3(6 - \cos^2 z) - z^2]$
I_6	$(-5 + 105y^2 - 315y^4 + 231y^6)/16$	$-210z^{-6}[99(3 + 2\cos^2 z)$ $- 6z^2(15 + 2\cos^2 z) + 2z^4]$

[4]A similar problem may be formulated also with respect to the end displacements u°, v°. See, however, the footnote[11].

appropriate to dv°/dy = given by (10), $\sigma_x^\circ = 0$, and of the eigenfunctions

$$I_n(x, y) = \Re \sum_k C_{nk} \Phi_k(x, y) \tag{29}$$

appropriate to u° = given by (11), $\tau^\circ = 0$, where

$$\Re \equiv \text{“real part of”}$$
$$\sum \equiv \sum_{z_2, z_4, \cdots} \quad \text{for} \quad n = \text{even}, \qquad \sum \equiv \sum_{z_3, z_5, \cdots} \quad \text{for} \quad n = \text{odd} \tag{30}$$

TABLE II. Formulas for the edge values of the stresses.

K_n	τ_n°
K_0	$-\dfrac{4}{\pi}\displaystyle\int_0^\infty k_{\sigma e}\, d\lambda$
K_1	$-\dfrac{4}{\pi}\displaystyle\int_0^\infty [k_{\sigma 0} + k_{\tau 0}]\, d\lambda/\lambda^2$
K_2	$\dfrac{8}{\pi}\displaystyle\int_0^\infty [2k_{\sigma e} - k_{\tau e}]\, d\lambda/\lambda^2$
K_3	$\dfrac{8}{\pi}\displaystyle\int_0^\infty [6k_{\sigma 0} + (6 + \lambda^2)k_{\tau 0}]\, d\lambda/\lambda^4$
K_4	$-\dfrac{16}{\pi}\displaystyle\int_0^\infty \left[(126 + 34\lambda^2)k_{\sigma e} - (84 + 3\lambda^2)k_{\tau e} + \dfrac{7}{2}\lambda^2 y(1 - y^2)\right] d\lambda/\lambda^4$
K_5	$-\dfrac{16}{\pi}\displaystyle\int_0^\infty \left[(270 + 18\lambda^2)k_{\sigma 0} + (270 + 78\lambda^2 + \lambda^4)k_{\tau 0} - \dfrac{3}{8}\lambda^4(1 - 6y^2 + 5y^4)\right] d\lambda/\lambda^6$
K_6	$\dfrac{64}{\pi}\displaystyle\int_0^\infty \Big[(5940 + 1998\lambda^2 + 26\lambda^4)k_{\sigma e} - (4455 + 342\lambda^2 + 2\lambda^4)k_{\tau e}$ $+ \dfrac{\lambda^2}{2} y(1 - y^2)\left\{495 - \dfrac{3}{8}\lambda^2(1 - 33y^2)\right\}\Big] d\lambda/\lambda^6$

J_n	σ_n°
J_2	$\dfrac{12}{\pi}\displaystyle\int_0^\infty j_{\sigma e}\, d\lambda/\lambda^2$
J_3	$\dfrac{60}{\pi}\displaystyle\int_0^\infty [j_{\sigma 0} + j_{\tau 0}]\, d\lambda/\lambda^4$
J_4	$\dfrac{60}{\pi}\displaystyle\int_0^\infty [(14 + 3\lambda^2)j_{\sigma e} - 7j_{\tau e}]\, d\lambda/\lambda^4$
J_5	$\dfrac{420}{\pi}\displaystyle\int_0^\infty [(18 + \lambda^2)j_{\sigma 0} + (18 + 4\lambda^2)j_{\tau 0}]\, d\lambda/\lambda^6$
J_6	$\dfrac{420}{\pi}\displaystyle\int_0^\infty \left[(297 + 90\lambda^2 + 2\lambda^4)j_{\sigma e} - (198 + 12\lambda^2)j_{\tau e} + \dfrac{33}{4}\lambda^2(1 - 3y^2)\right] d\lambda/\lambda^6$

and summation extends over the first quadrant roots of (26). The two procedures, one due to the author, the other based on a method of R. C. T. Smith, by which the coefficients C_{nk} may be determined, are described in Secs. 3 and 4, respectively.

Anticipating the results of Sec. 3, and utilizing the functions $k(\lambda, y)$, $j(\lambda, y)$ listed in Appendix I, the edge stresses τ_n° appropriate to K_n and the edge stresses σ_n° appropriate to J_n are found to have the integral representations listed in Table II. We give these expressions up to $n = 6$. Numerical integration, performed as in [4], then leads to the edge values of the stresses $\tau_0^\circ, \tau_1^\circ, \cdots, \sigma_4^\circ$ displayed in Table III. The scalar products $\langle\tau_n^\circ\tau_k^\circ\rangle$, $\langle\sigma_n^\circ\sigma_k^\circ\rangle$, obtained by numerical integration over the tabulated values, are listed in Table IVa[5].

TABLE III. Edge values of the stresses.

$y =$	0	.2	.4	.6	.8	.9	.95	1.0⁻
τ_0°	0	−.10038	−.20638	−.32458	−.46371	−.54490	−.58929	−.63662
τ_1°	.18917	.17263	.11981	.01867	−.16549	−.32227	−.43786	−.63662
τ_2°	0	−.14339	−.27328	−.36970	−.38606	−.31681	−.22945	0
τ_3°	.15800	.13177	.05739	−.05029	−.15417	−.16970	−.14317	0
τ_4°	0	−.3644	−.5143	−.29678	.27376	.54880	.56353	0
σ_2°	.58820	.53678	.37249	.05747	−.51596	−1.0022	−1.3588	−1.9665
σ_3°	0	.47538	.81663	.83568	.1329	−.86634	−1.7833	−3.7827
σ_4°	−.87136	−.57794	.19518	1.0672	1.0413	−.25672	−1.8781	−6.3727

TABLE IV. (a) *Scalar products of* $\tau_n^\circ, \sigma_n^\circ$. (b) *The expansion coefficients* b_{nk}, a_{nk} *of* t_n, s_n *in terms of* $\tau_k^\circ, \sigma_k^\circ$, *Eqs.* (21), (25)*.

nk	$\langle\tau_n^\circ\tau_k^\circ\rangle$	$\langle\sigma_n^\circ\sigma_k^\circ\rangle$	b_{nk}	a_{nk}
22	.1598	.7912	2.501	−1.124
33	.02726	1.590	6.056	−.7931
44	.2848	2.722	−1.939	−.6279
42	.05470	.3836	.6634	.3044

The orthogonalization processes (22), (25), finally lead, in accordance with (23), (24), to the expansion coefficients b_{nk}, a_{nk} of H_n, G_n in terms of K_k, J_k, as shown in Table IVb. The boundary stresses t_2, t_3, t_4 produced by H_n and the boundary stresses s_2, s_3, s_4 produced by G_n are plotted in Figs. 1 and 2. For comparison, the corresponding distributions based on orthonormalized self-equilibrating polynomials are also shown. (It will be recalled that the $f_k(y)$ polynomials are orthogonal, the $f_k'(y)$, $f_k''(y)$ polynomials are not. They may be readily orthonormalized into functions

$$T_n = \sum_2^n B_{nk} f_k', \qquad S_n = \sum_2^n A_{nk} f_k'' \tag{31}$$

as shown in [2].) Note that the disagreement of the two sets of curves is not a measure of the inaccuracy of the variational approach, but reflects merely a rotation in function space from one set of orthogonal axes to another.

*The last two digits of the entries of Tables IVa,b are uncertain. See footnote[5].

[5]Because this numerical integration was carried out over the sparsely determined values, at $y = 0$, .2, .4, .6, .8, .9, .95, 1.0, the entries of the present Tables IVa,b must be regarded as preliminary estimates. However, a more accurate determination would require a tremendous investment in time and personnel; these are not available to the author.

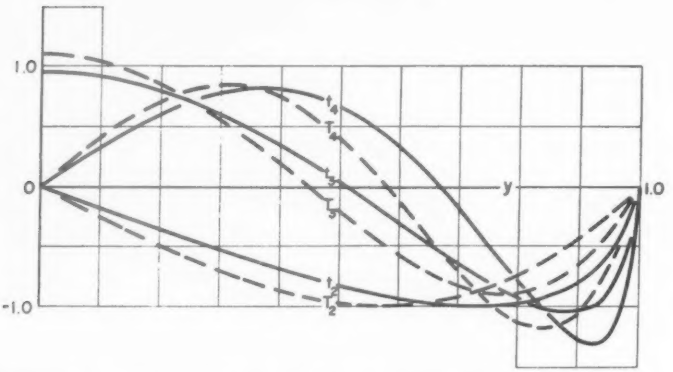

FIG. 1. The first three orthonormal boundary shear tractions. Full lines:

$$t_n(y) \equiv -H^{\circ}_{n,xy} \equiv -\sum_{i=2}^{n} b_{ni} K^{\circ}_{i,xy} ;$$

dashed lines:

$$T_n(y) \equiv \sum_{i=2}^{n} B_{ni} f_i' .$$

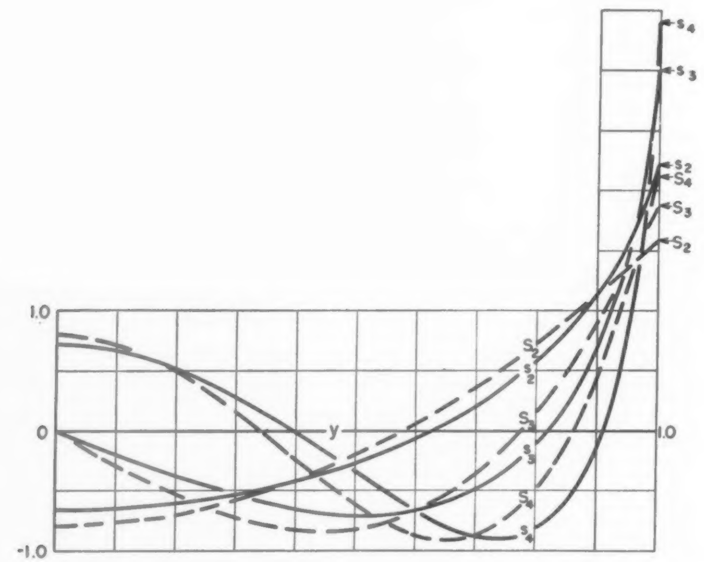

FIG. 2. The first three orthonormal boundary normal tractions. Full lines:

$$s_n(y) \equiv G^{\circ}_{n,yy} \equiv \sum_{i=2}^{n} a_{ni} J^{\circ}_{i,yy} ;$$

dashed lines:

$$S_n(y) \equiv \sum_{i=2}^{n} A_{ni} f_i'' .$$

3. The method of analytic continuation. This method of approach was discovered in a somewhat accidental manner. In [4] we investigated the problem of the stress state in an *infinite* strip $-\infty \le x \le +\infty$, $-1 \le y \le +1$, occasioned by the temperature distribution

$$\vartheta(x, y) = \begin{cases} T & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x > 0 \end{cases} \tag{32}$$

or, what amounts essentially to the same thing, by the edge tractions

$$\tau^* = 0, \qquad \sigma_y^* = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \tag{33}$$

and found, incidental to the solution of the thermal stress problem, also the solution $\frac{1}{2} K_0(x, y)$ of the *semi-infinite* strip for the mixed edge conditions

$$\sigma_x^\circ = 0, \qquad dv^\circ/dy = 1/2. \tag{34}$$

Similarly, solution in [7] of the thermal stress problem of the *infinite* strip for

$$\vartheta(x, y) = \begin{cases} -T'x & x < 0 \\ 0 & x > 0 \end{cases} \tag{35}$$

or, what amounts essentially to the same thing, for the traction distribution

$$\tau^* = 0, \qquad \sigma_y^* = \begin{cases} | x | & x < 0 \\ 0 & x > 0 \end{cases} \tag{36}$$

provided also solution $J_2/6$ of the *semi-infinite* strip problem for the mixed edge conditions

$$\tau^\circ = 0, \qquad u^\circ = (3y^2 - 1)/12. \tag{37}$$

It was then immediately obvious that all end problems of the semi-infinite strip should be reducible to problems of the (doubly) infinite strip. But instead of seeking determination, from a pair of integral equations, of complicated unknown tractions σ_y^*, τ^* applied to the left half, $x < 0$, of the horizontal edges of an infinite strip, in terms of the given values at $x = 0$ (dv°/dy = given, $\sigma_x^\circ = 0$; or u° = given, $\tau^\circ = 0$), one should be able to arrive at the tractions σ_y^*, τ^* by inspection. This is the very program that was carried out in [4], [7] and [2], and led to the determination of the stress functions K_0, J_2, and K_1, J_3. The singularities of these four functions at $x = -\infty$ are, however, not quite severe enough to illustrate fully the general approach. For this reason we determine below the stress function K_4 of the (doubly) infinite strip, appropriate to the boundary conditions

$$\begin{aligned} &\text{for} \quad x > 0: \quad \sigma_y^* = \tau^* = 0 \\ &\text{for} \quad x = 0: \quad \sigma_x^\circ = 0, \qquad dv^\circ/dy = 1 - 8y^2 + 7y^4. \end{aligned} \tag{38}$$

We shall find that K_4 has the form

$$K_4 = \frac{63}{2} K_{4\sigma} - 42K_{4\tau} - 34K_{2\sigma} - 6K_{2\tau}\,, \tag{39}$$

where the functions $K_{2\sigma}, \cdots, K_{4\tau}$ of x and y are defined, in terms of

$$\Delta_e A_e = sh\lambda + \lambda ch\lambda, \qquad \Delta_e B_e = -\lambda sh\lambda, \qquad \Delta_e C_e = sh\lambda, \qquad \Delta_e D_e = -ch\lambda, \tag{40}$$

(see the Appendix for Δ_e) as follows:[6]

$$K_{2\sigma} = -\frac{2^4}{\pi}\int_0^\infty \left[Ach\lambda y + Bysh\lambda y - \frac{1}{2}\right]\frac{\sin \lambda x}{\lambda^5}\,d\lambda + 4\left\{\frac{x^4}{4!} - \frac{2}{\pi}\int_0^\infty \left(\sin\lambda x - \lambda x + \frac{\lambda^3 x^3}{3!}\right)\frac{d\lambda}{\lambda^5}\right\}, \tag{41a}$$

$$K_{2\tau} = \frac{2^3}{\pi}\int_0^\infty \left[Cch\lambda y + Dysh\lambda y - \frac{1}{4}(1 - y^2)\right]\frac{\sin\lambda x}{\lambda^3}\,d\lambda + (1 - y^2)\left\{\frac{x^2}{2!} + \frac{2}{\pi}\int_0^\infty (\sin\lambda x - \lambda x)\frac{d\lambda}{\lambda^3}\right\}, \tag{41b}$$

$$\begin{aligned} K_{4\sigma} = \frac{2^6}{\pi}\int_0^\infty &\left[Ach\lambda y + Bysh\lambda y - \frac{1}{2} + \frac{\lambda^4}{48}(1 - y^2)^2\right]\frac{\sin\lambda x}{\lambda^7}\,d\lambda \\ &+ 16\left\{\frac{x^6}{6!} + \frac{2}{\pi}\int_0^\infty \left(\sin\lambda x - \lambda x + \frac{\lambda^3x^3}{3!} - \frac{\lambda^5x^5}{5!}\right)\frac{d\lambda}{\lambda^7}\right\} \\ &- \frac{2}{3}(1 - y^2)^2\left\{\frac{x^2}{2!} + \frac{2}{\pi}\int_0^\infty (\sin\lambda x - \lambda x)\frac{d\lambda}{\lambda^3}\right\}, \end{aligned} \tag{41c}$$

$$\begin{aligned} K_{4\tau} = \frac{2^5}{\pi}\int_0^\infty &\left[Cch\lambda y + Dysh\lambda y - \frac{1}{4}(1 - y^2) + \frac{\lambda^2}{24}(1 - y^2)^2\right]\frac{\sin\lambda x}{\lambda^5}\,d\lambda \\ &- 4(1 - y^2)\left\{\frac{x^4}{4!} - \frac{2}{\pi}\int_0^\infty \left(\sin\lambda x - \lambda x + \frac{\lambda^3x^3}{3!}\right)\frac{d\lambda}{\lambda^5}\right\} \\ &- \frac{2}{3}(1 - y^2)^2\left\{\frac{x^2}{2!} + \frac{2}{\pi}\int_0^\infty (\sin\lambda x - \lambda x)\frac{d\lambda}{\lambda^3}\right\}. \end{aligned} \tag{41d}$$

We furthermore abbreviate

$$\begin{aligned} \mathcal{K}_{2\sigma} &= K_{2\sigma} - x^2y^2/2, \qquad \mathcal{K}_{2\tau} = K_{2\tau} + x^4/6, \\ \mathcal{K}_{4\sigma} &= K_{4\sigma} - \frac{2x^4y^2 - 4x^2y^4}{9} + \frac{2x^4 - 12x^2y^2}{9}, \\ \mathcal{K}_{4\tau} &= K_{4\tau} - \frac{x^6 - 15x^2y^4}{60} - \frac{x^2y^2}{6}. \end{aligned} \tag{42}$$

We start the analysis by considering, for the time being, the following problem. Find the stress function K of the infinite strip subject to boundary tractions

[6] In what follows we omit the subscripts e of A, B, C, D for the sake of simpler notation. The coefficients A_0, B_0, C_0, D_0 which arise in the K_n, J_n expressions when n is odd, are given in [2], Eq. (13).

$$\tau^* = 0, \qquad \sigma_y^* = \begin{cases} 2^5x^4/4! & x < 0 \\ 0 & x > 0 \end{cases} \tag{43}$$

or what amounts to the same, find $K(x, y)$ so that[7]

$$\begin{aligned} K^*_{,xy} &= 0, \\ 2^{-4}K^*_{,xx} &= \frac{x^4}{4!} - \frac{2}{\pi}\int_0^\infty \left(\sin\lambda x - \lambda x + \frac{\lambda^3x^3}{3!}\right)\frac{d\lambda}{\lambda^5}. \end{aligned} \tag{44}$$

It is easy to see that insofar as the first term in the integrand of (44) is concerned

$$\frac{64}{\pi}\int_0^\infty \left[Ach\lambda y + Bysh\lambda y\right]\frac{\sin\lambda x}{\lambda^7}\,d\lambda \tag{45}$$

is the solution, $A(\lambda)$, $B(\lambda)$ being given by (40). But (45) is not a satisfactory expression since it diverges at $\lambda = 0$, moreover no terms are given in (45) which provide the x, $x^3/3!$ terms of the integrand of (44). We first take care of the singularity of the expression (45). Noting that

$$\begin{aligned} Ach\lambda y + Bysh\lambda y = \frac{1}{2} + \lambda^4(1-y^2)^2\Big[-\frac{1}{2\cdot 4!} &+ \frac{\lambda^2}{6!}(3-y^2) \\ &- \frac{\lambda^4}{6\cdot 8!}(41 - 66y^2 + 9y^4) + \cdots\Big], \end{aligned} \tag{46}$$

we subtract out the singular part from (45), as shown in the expression (41c). These subtracted out terms are nonbiharmonic, so we must add them back in, and this is done in the braces { } of $K_{4\sigma}$. To compensate furthermore for the singularities introduced by the terms $\sin\lambda x/\lambda^7$ etc. into the braces, we add, under the integral signs of these braces, such terms λx, $\lambda^3x^3/3!$, etc., that the singularity be removed. The addition of $x^6/6!$ and $x^2/2!$ in the braces outside of the integral sign is finally made in order to insure the condition

$$K^*_{4\sigma,xx} = 0, \qquad K^*_{4\sigma,xy} = 0 \quad \text{for} \quad x > 0. \tag{47}$$

For $x < 0$ there follows then from (41c)

$$K^*_{4\sigma,xy} = 0, \qquad K^*_{4\sigma,xx} = 2^5x^4/4! \tag{48}$$

Thus, we succeeded in constructing a function, $K_{4\sigma}$, which satisfies the boundary conditions (43), (44). Unfortunately, $K_{4\sigma}$ suffers from the defect that it is (because of the presence of terms in the braces which are outside of the integral sign) not biharmonic. This defect can be corrected by addition of suitable terms, as in (42c), to make it a biharmonic function $\mathcal{K}_{4\sigma}$; but then the conditions (47), (48) become (mildly) violated.

So we start out on a new tack. We consider the problem of finding a stress function $K_{4\tau}$ which gives

$$\sigma_y^* = 0, \qquad \tau^* = \begin{cases} 2x^3y & x < 0 \\ 0 & x > 0 \end{cases} \tag{49}$$

[7]Compare Eqs. (11) of [2].

or, what amounts to the same thing, finding a biharmonic function which assumes the boundary values

$$K^*_{4r,xx} = 0,$$
$$K^*_{4r,xy} = 8y\left\{\frac{x^3}{3!} - \frac{2}{\pi}\int_0^\infty \left(\cos\lambda x - 1 + \frac{x^2\lambda^2}{2}\right)\frac{d\lambda}{\lambda^4}\right\}. \tag{50}$$

Proceeding as before, we are first lead to the C and D terms in K_{4r}, and then, noting that

$$Cch\lambda y + Dysh\lambda y = \frac{1}{4}(1 - y^2) + \frac{\lambda^2}{4}(1 - y^2)^2\left[-\frac{1}{3!} - \frac{\lambda^2}{3\cdot 5!}(1 + 3y^2) + \cdots\right] \tag{51}$$

to the complete expression K_{4r} of (41d) which satisfies the boundary conditions

$$\begin{aligned} &\text{for } x > 0: && K^*_{4r,xx} = K^*_{4r,xy} = 0 \\ &\text{for } x < 0: && K^*_{4r,xx} = 0, \qquad K^*_{4r,xy} = 8x^3y/3 \end{aligned} \tag{52}$$

but is nonbiharmonic, and the modified expression $\mathcal{K}_{4r}$ of (42d) which does not satisfy the condition (52) but is biharmonic. Note, however, that by taking the combination

$$K_{4\sigma} - \frac{4}{3}K_{4r} \tag{53}$$

we eliminate the nonbiharmonicity of the 6th degree terms x^6, x^4y^2, x^2y^4 in the combination, and by adding on to (53) a suitable combination of $K_{2\sigma}$, K_{2r} (in the present instance, $8K_{2r}/9$) we eliminate the nonbiharmonicity also of the 4th degree terms. Thus,

$$\mathcal{K} \equiv K_{4\sigma} - \frac{4}{3}K_{4r} + \frac{8}{9}K_{2r} \tag{54}$$

is biharmonic and it satisfies the boundary conditions[8]

$$\mathcal{K}^*_{,xx} = \begin{cases} 0 & x > 0 \\ 2^5x^4/4! & x < 0, \end{cases} \tag{55a,b}$$

$$\mathcal{K}^*_{,xy} = \begin{cases} 0 & x > 0 \\ -32(x^3 + x)/9 & x < 0, \end{cases} \tag{55c,d}$$

$$\mathcal{K}^\circ_{,yy} \equiv \sigma^\circ_x = 0, \qquad \mathcal{K}^\circ_{,xx} \equiv dv^\circ/dy = \frac{2}{9}(1 - y^2)(5 - y^2). \tag{56a,b}$$

Our last step is a rather minor one. We take the combination

$$K_4 = \frac{63}{2}\mathcal{K} - 34K_2 \qquad (K_2 = K_{2\sigma} + K_{2r}) \tag{57}$$

as stated earlier, in Eq. (39). In this fashion we modify our tractions to

$$K^*_{4,xx} = \begin{cases} 0 & x > 0 \\ 42x^4 - 136x^2 & x < 0, \end{cases} \tag{58a}$$

[8]The boundary conditions (55b,d) are just as suitable for our purposes as were the proposed conditions (43). Equations (43) were not an objective, but merely a starting point. The objective is determination of some biharmonic function $K(x, y)$ which satisfies (55a,c), (56a) and leads to some 4th degree polynomial of the type (56b).

$$K^*_{4,xy} = \begin{cases} 0 & x > 0 \\ -112x^3 + 24x & x < 0, \end{cases} \tag{58b}$$

$$K^\circ_{4,yy} = 0, \qquad K^\circ_{4,xx} = 1 - 8y^2 + 7y^4, \tag{58c,d}$$

and thus achieve that the dv°/dy distribution belonging to K_4 is orthogonal to the dv°/dy distribution belonging to K_2 . This facilitates expansion of a given dv°/dy into $K^\circ_{n,xx}$ functions. Higher K_n functions are constructed similarly.

We have thus found that the mixed edge value problem of the semi-infinite strip

$$\sigma^\circ_x = 0, \qquad dv^\circ/dy = n\text{th degree polynomial in } y \tag{59}$$

may be converted into the problem of determining suitable distributions σ^*_y , τ^* which are 0 for $x > 0$, and for $x < 0$ they are n-degree and $(n - 1)$-degree polynomials, respectively, when n = even, and $(n - 1)$-degree and n-degree polynomials, respectively, when n = odd. Furthermore, the stress functions of these σ^*_y , τ^* distributions may be derived in a systematic, though very tedious, manner. So one really does not have to solve for them, but merely construct their expressions. Nevertheless, for large n the Fourier integral representations of $K_n(x, y)$, $J_n(x, y)$ become unmanageably cumbersome. For the region $x \geq 0$ the braces of the $K_{n\sigma}$, $K_{n\tau}$, $J_{n\sigma}$, $J_{n\tau}$ expressions [see Eqs. (41)] vanish, and what remains may be converted by contour integration into rather "simple looking" infinite series, as shown in [4], [7], [2]. These are the expressions displayed in Eqs. (28) and in the first part of Table I[9].

The mixed end problem of the semi-infinite strip

$$\tau^\circ = 0, \; u^\circ = n\text{th degree polynomial in } y \tag{60}$$

leads in a completely similar manner to distributions σ^*_y , τ^* which are zero for $x > 0$, and for $x < 0$ they are $n - 1$, $n - 2$ degree polynomials when n = even, and $n - 2$, $n - 1$ degree polynomials when n = odd. The biharmonic eigenfunction expansions of the stress functions, corresponding to various distributions u°, are displayed in Eq. (29) and in the second part of Table I.

Once we have the integral expressions of K_n , J_n we may also obtain, by differentiation, the boundary values

$$\tau^\circ_n \equiv -K^\circ_{n,xy} , \qquad \sigma^\circ_n \equiv J^\circ_{n,yy} . \tag{61}$$

Note particularly that these stress boundary values involve only terms from the brackets of the type (41) expressions and no terms from the braces. (The terms in the braces are required, as shown in [2], for determining the unknown displacement—$u^\circ(y)$ in the case of K, $dv^\circ(y)/dy$ in the case of J—along edge $x = 0$.) By orthonormalizing the functions τ°_n , σ°_n into functions t_n , s_n , as outlined in the Introduction, we are finally led to the solutions H_n , G_n of the "pure" end problem.

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[9]The minute one tries to use the series in numerical work, the simplicity is replaced by extreme complexity. The writer is therefore inclined to believe that the approximate method of "self-equilibrating polynomials", developed in [3], and well established in regard to simplicity and adequate reliability in [4], [7], [8], [9], [2], [14] is likely to remain the favored technique in the solution of *practical* problems, at least until the time when tables of the functions K_n , J_n and their derivatives become available. (An extension of the self-equilibrating function method to stress problems in polar coordinates is outlined in [13].)

4. The expansion formula of R. C. T. Smith. While in the previous section we carried out the program of solving the mixed and the pure end problems of the semi-infinite strip, we did more than was proposed, we obtained, in addition, also the distributions σ_y^*, τ^* which give rise to the boundary values σ_x°, τ°. This latter result, while of profound mathematical interest is, nevertheless, not germane to the original problem. To obtain σ_y^*, τ^*, i.e., to obtain expressions like Eqs. (39), (41), we had to pay a very high price in amount of labor.

The great amount of labor resulted from the necessity for determining tractions at $x = -\infty$ which balance certain infinite resultants of the σ_y^*, τ^* distributions. These contributions at $-\infty$ were brought into rather sharp focus in [2][10]. However, in our immediate objective we are not interested in what happens at $x = -\infty$, nor, indeed, are we interested in what happens at $x < 0$. We are interested only in what happens to the right of $x = 0$. It is, therefore, very desirable to find an alternate method, which bypasses the determination of σ_y^*, τ^*, and leads directly to the eigenfunction expansions of the K_n, J_n, H_n, G_n sets. Such a direct route is, indeed, provided in the excellent work of R. C. T. Smith [10][11].

Smith in his paper has shown (using a different notation) that the boundary value problem of the semi-infinite strip

$$\begin{gathered} \nabla^4\Phi = 0, \qquad \Phi^*_{,xx} = \Phi^*_{,xy} = 0, \\ \Phi^\circ_{,xx} = \text{given}, \qquad \Phi^\circ_{,yy} = \text{given}, \end{gathered} \tag{62}$$

may be solved in the form of a series in the Fadle-Papkovitch functions (27),

$$\Phi = \mathcal{R} \sum C_k \Phi_k , \tag{63}$$

where

$$C_k = \begin{cases} (z_k \sin^3 z_k/2 \cos^3 z_k) \int & (k = \text{even}), \\ -(z_k \cos^3 z_k/2 \sin^3 z_k) \int & (k = \text{odd}), \end{cases} \tag{64}$$

$$\int \equiv \int_{-1}^{+1} [\Phi^\circ_{k,yy} + 2z_k^2\Phi^\circ_k , z_k^2\Phi^\circ_k]\cdot\begin{bmatrix} -\Phi^\circ_{,yy} \\ \Phi^\circ_{,xx} + 2\Phi^\circ_{,yy} \end{bmatrix} dy. \tag{65}$$

In particular, for the cases of

$$\sigma_x^\circ = 0, \qquad \Phi^\circ_{,xx} = dv^\circ/dy = \text{given}, \tag{66a}$$

$$\tau^\circ = 0, \qquad \Phi^\circ_{,yy} = u^\circ = \text{given}, \tag{66b}$$

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[10] Ref. [2] gives many details and side lights which could not be compressed into the present paper.

[11] The author is indebted to Professor Eric Reissner for calling attention to Smith's work. The papers [4], [7], and the first version of [2] were prepared without knowledge of Smith's paper. The original program, and many of the calculations of the present paper were also carried out without the benefit of familiarity with Smith's results. However, the functions J_4, J_5, J_6, K_3, K_4, K_6 were obtained on the basis of Smith's expansion formula.

formula (65) reduces to

$$\int = z_k^2 \int_{-1}^{+1} \Phi_k^\circ(dv^\circ/dy)\, dy, \qquad \int = -\int_{-1}^{+1} \Phi_{k.yy}^\circ u^\circ\, dy, \tag{67a,b}$$

respectively.

For the case (66a), (67a) the function Φ of (63) is the stress function (we refer to it as function K); for the case (66b), (67b) the x-derivative of Φ

$$J = \Phi_{,x} = \mathcal{R} \sum C_k \Phi_{k.x} \tag{68}$$

is the stress function. Letting

$$l^{(i)} \equiv l(l-1) \cdots (l-i+1) \tag{69}$$

and noting

$$\int_{-1}^{+1} \Phi_k^\circ y^l\, dy = \frac{1}{(l+2)(l+1)} \int_{-1}^{+1} \Phi_{k.yy}^\circ y^{l+2}\, dy$$

$$= -\frac{4 \cos z_k}{z_k^{l+4}} \{z_k^l - 2l^{(2)} z_k^{l-2} + 3l^{(4)} z_k^{l-4} - 4l^{(6)} z_k^{l-6} + \cdots - \cos^2 z_k [l z_k^{l-2} - 2l^{(3)} z_k^{l-4} + 3l^{(5)} z_k^{l-6} - \cdots]\} \tag{70a}$$

for $k, l =$ even,

$$\int_{-1}^{+1} \Phi_k^\circ y^l\, dy = \frac{1}{(l+2)(l+1)} \int_{-1}^{+1} \Phi_{k.yy}^\circ y^{l+2}\, dy$$

$$= -\frac{4}{z_k^{l+4} \cos z_k} \{z_k^{l+1} - 2l^{(2)} z_k^{l-1} + 3l^{(4)} z_k^{l-3} - \cdots - \sin^2 z_k [l z_k^{l-1} - 2l^{(3)} z_k^{l-3} + 3l^{(5)} z_k^{l-5} - \cdots]\} \tag{70b}$$

for $k, l =$ odd, we determine the integrals listed in Table V and thereby also the coefficients C_k of expansions (63), (68), when dv°/dy, u° are powers of y.

Note that the expansions obtained will, in general, not converge for $x = 0$, since the conditions of self-equilibration are not, in general, satisfied by the distributions y^l. However, the expansions are merely the building blocks which make up the complete self-equilibrating stress functions of Tables Ia,b; the latter expansions, as Smith has shown, do converge. Smith's procedure therefore permits complete solution of the two *mixed* end problems. In order to go beyond the mixed problem and resolve the *pure* end problem, we have to determine the edge values of the τ_n°, σ_n° distributions, in the form given in Table II. However, even this particular representation can be arrived at by an extension of the Smith technique (without prior determination of the distributions σ_v^*, τ^* which give rise to it) by merely retracing the steps employed in the usual application of the residue theorem. Starting out with, e.g.,

$$-\tau_2^\circ = K_{2.xy}^\circ = \mathcal{R} \sum_k \frac{4(2 + \cos^2 z_k) \sin^2 z_k}{z_k^2 \cos^3 z_k} \left[\left(1 + \frac{\cot z_k}{z_k}\right) \sin z_k y + y \cot z_k \cos z_k y\right]$$

$$\equiv \mathcal{R} \sum_k \frac{\mathcal{A}(z_k) \sin z_k y + \mathcal{B}(z_k) y \cos z_k y}{4 \cos^2 z_k}, \tag{71}$$

$$\mathcal{A}(z_k) = -\frac{16}{z_k^2}(3 \cos z_k + z_k \sin z_k), \qquad \mathcal{B}(z_k) = \frac{16}{z_k^2}(2 \sin z_k - z_k \cos z_k),$$

TABLE V. Integrals of $\Phi_k^\circ y^l$. (Subscript k of z is omitted for the sake of simpler notation.) The upper group pertains to k = even, the lower group to k = odd. The integrals vanish when k + l = odd.

l	$z^2 \int_{-1}^{1} \Phi_k^\circ y^l \, dy = \dfrac{z^2}{(l+2)(l+1)} \int_{-1}^{1} \Phi_{k.yy}^\circ y^{l+2} \, dy$
0	$-4 \cos z/z^2$
2	$(-4 \cos z/z^4)[z^2 - 4 - 2 \cos^2 z]$
4	$(-4 \cos z/z^6)[z^4 - 24z^2 + 72 - \cos^2 z(4z^2 - 48)]$
6	$(-4 \cos z/z^8)[z^6 - 60z^4 + 1080z^2 - 2880 - \cos^2 z(6z^4 - 240z^2 + 2160)]$
8	$(-4 \cos z/z^{10})[z^8 - 112z^6 + 5040z^4 - 80640z^2 + 201600$ $- \cos^2 z(8z^6 - 672z^4 + 20160z^2 - 161280)]$
1	$(-4/z^3 \cos z)[z^2 - \sin^2 z]$
3	$(-4/z^5 \cos z)[z^4 - 12z^2 - \sin^2 z(3z^2 - 12)]$
5	$(-4/z^7 \cos z)[z^6 - 40z^4 + 360z^2 - \sin^2 z(5z^4 - 120z^2 + 360)]$
7	$(-4/z^9 \cos z)[z^8 - 84z^6 + 2520z^4 - 20160z^2 - \sin^2 z(7z^6 - 420z^4 + 7560z^2 - 20160)]$
9	$(-4/z^{11} \cos z)[z^{10} - 144z^8 + 9072z^6 - 241920z^4 + 1814400z^2$ $- \sin^2 z(9z^8 - 1008z^6 + 45360z^4 - 725760z^2 + 1814400)]$

we may write, on noting

$$\frac{d}{dz_k}(\sin 2z_k + 2z_k) = 2(1 + \cos 2z_k) = 4 \cos^2 z_k , \tag{72}$$

the integral representation $(0 < \epsilon < \mathcal{R}(z_2))$

$$-\tau_2^\circ = \frac{1/2}{2\pi i} \int_{\epsilon + i\infty}^{\epsilon - i\infty} \frac{\mathcal{A}(z) \sin zy + \mathcal{B}(z) y \cos zy}{\sin 2z + 2z} \, dz. \tag{73}$$

Introducing

$$\lambda = -iz \tag{74}$$

we finally convert (73) to

$$-\tau_2^\circ = \frac{1}{\pi} \int_0^\infty \left[\frac{\mathcal{U}(\lambda) ch\lambda y + \mathcal{V}(\lambda) y sh\lambda y}{sh 2\lambda + 2\lambda} + \mathcal{C}(\lambda, y) \right] d\lambda, \tag{75}$$

where $\mathcal{C}(\lambda, y)$ is so determined that the infinities of the $\mathcal{U}$, $\mathcal{V}$ terms are compensated. This leads us back to our τ_2^0 expression in Table II. (This is the way the edge distributions σ_4°, σ_5°, σ_6°, τ_3°, τ_5°, τ_6° of Table II were determined.)

It may be added that the approach of Sec. 3, while evidently superfluous in the final establishment of Tables I to V, is by no means superfluous in the deeper insight it has given for the solution of the end problem. In fact, it was the very search for the

distributions σ_y^*, τ^* which give prescribed σ_x°, τ° that provided the guiding idea which lead to the solution of the pure end problem of the semi-infinite strip[12].

Acknowledgement. The author is very much indebted to his colleague, Mrs. J. S. Born, for the tremendous task of computing the Tables I to V and the (not shown) power series expansions of the functions in the Appendix, and for checking all derivations.

Appendix. The functions $\Delta(\lambda)$, $k(\lambda, y)$, $j(\lambda, y)$[13].

$$\Delta_e = sh2\lambda + 2\lambda, \qquad \Delta_0 = sh2\lambda - 2\lambda,$$

$$\Delta_e k_{\sigma e} = ch\lambda sh\lambda y - sh\lambda \cdot ych\lambda y,$$

$$\Delta_e k_{\tau e} = (\lambda sh\lambda - ch\lambda)sh\lambda y - \lambda ch\lambda \cdot ych\lambda y + \tfrac{1}{2}y\Delta_e\,,$$

$$\Delta_0 k_{\sigma 0} = \lambda^2 sh\lambda ch\lambda y - \lambda^2 ch\lambda \cdot ysh\lambda y - \tfrac{3}{4}(1 - y^2)\Delta_0\,,$$

$$\Delta_0 k_{\tau 0} = (sh\lambda - \lambda ch\lambda)ch\lambda y + \lambda sh\lambda \cdot ysh\lambda y + \tfrac{1}{4}(1 - 3y^2)\Delta_0\,,$$

$$\Delta_e j_{\sigma e} = (\lambda ch\lambda - sh\lambda)ch\lambda y - \lambda sh\lambda \cdot ysh\lambda y,$$

$$\Delta_e j_{\tau e} = (\lambda^2 sh\lambda - 2\lambda ch\lambda)ch\lambda y - \lambda^2 ch\lambda \cdot ysh\lambda y + \tfrac{1}{2}\Delta_e\,,$$

$$\Delta_0 j_{\sigma 0} = (\lambda^3 sh\lambda - \lambda^2 ch\lambda)sh\lambda y - \lambda^3 ch\lambda \cdot ych\lambda y + \tfrac{3}{2}y\Delta_0\,,$$

$$\Delta_0 j_{\tau 0} = (2\lambda sh\lambda - \lambda^2 ch\lambda)sh\lambda y + \lambda^2 sh\lambda \cdot ych\lambda y - \tfrac{3}{2}y\Delta_0\,.$$

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[12]There exists a second type of pure end problem which is of interest, namely, the one where on the edge $x = 0$ both displacements u° and v° are specified. Since this group of problems is usually characterized by infinite stresses in the corners $(x, y) = (0, \pm 1)$, the present scheme of infinite expansions cannot be adopted without first isolating the infinity or taking other precautionary measures, see [11], [12].

[13]The writer shall be glad to pass on, to those interested, power series expansions of the $k(\lambda, y)$, $j(\lambda, y)$ functions (these are needed for evaluating the $\lambda = 0$ values of the integrands of Table II) as well as tabulated values of the functions for $\lambda = 0, .4, .8, \ldots, 6.4$, at $y = 0, .2, .4, .6, .8, .9, .95, 1.0$.

BOOK REVIEWS

(*Continued from p. 64*)

Les fonctions orthogonales dans les problemes aux limites de la physique mathematique. By Theodore Vogel. Renseignements et Vente au Service des Publications du CNRS, 45 Rue d'ulm, Paris, 1953. iv + 191 pp. $3.50.

This is a short compendium on orthogonal functions and their applications in solving boundary value problems in mathematical physics. The book is intended for engineers, physicists and applied mathematicians. It is not a textbook, but rather an exposition of results and methods. Nevertheless the book is very readable, and while most theory is stated without proof, the main ideas are brought out clearly and forcefully, and the reader gets a good picture of the subject. An extensive but selective bibliography makes the book particularly valuable. An unusual feature of a book directed primarily toward applications is the high standard of rigor in stating results. In particular, the Lebesgue integration theory is assumed and used.

The Table of Contents gives an idea of the large amount of information which is contained in this slim but substantial volume.

Chapter I. Orthogonal functions and differential systems.
1] Closed sequences of orthogonal functions.
2] Differential systems and orthogonal functions.
3] Closure of certain sequences of Green functions.
4] Perturbed systems.
Chapter II. Study of some particular closed sequences.
5] Trigonometric functions and Fourier series.
6] Bessel functions.
7] Legendre functions.
8] Orthogonal polynomials numerals.
9] Mathieu functions.
Chapter III. Examples of applications.
10] Separation of variables.
11] Boundary value problems for the wave equation.
12] Boundary value problems for the Laplace equation.
13] The heat equation.
14] Equations of the fourth order.
15] Perturbed problems.

L. Bers

Numerical methods. By A. D. Booth. Butterworths Scientific Publications, London, 1955. vii + 195 pp. $6.00.

The orientation of the book is best indicated by the following passages from the preface. "Some existing books on numerical analysis lay much stress upon the detailed form in which a given procedure is to be laid out; ··· such concentration upon actual numbers obscures the underlying mathematical basis upon which the work rests, and such tabulations are almost entirely absent from this book." "The classical methods of hand calculation are, to a greater or less extent, unsuitable for the modern machines, and only by having a thorough knowledge of the underlying mathematical principles, is the programmer likely to make effective use of the new tools." The scope of the book is shown by the following chapter headings, each parenthesis containing the number of pages devoted to the topic: The nature and purpose of numerical analysis (6)—Tabulations and differences (5)—Interpolation (16)—Numerical differentiation and integration (23)—The summation of series (5)—The solution of ordinary differential equations (16)—Simultaneous linear equations (38)—Partial differential equations (32)—Non-linear algebraic equations (19)—Approximating functions (9)—Fourier synthesis and analysis (10)—Integral equations (8). Since the space does not permit anything like an exhaustive treatment of these topics, the author restricts himself to discussing a small selection of proved methods. The exposition is clear and easy to follow. The illustrative examples frequently have unusual features that challenge the numerical analyst.

W. Prager

(*Continued on p. 112*)

THE INTERACTION OF AN ACOUSTIC WAVE AND AN ELASTIC SPHERICAL SHELL*

BY

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Abstract. The effect of the impact of a plane pressure wave on an elastic spherical shell is considered, taking into account both the incident and diffracted waves. An infinite series (mode) solution is used and numerical results are obtained for the case of a steel shell in water. It is shown that a simple quasi-static system can be used to find a very good approximation for the stress. The total acceleration on initial impact is found exactly and seen to differ markedly from the accelerations associated with the lower vibrational modes.

1. Introduction. This problem was suggested by G. F. Carrier in consequence of work previously done by him on a related problem [1][1] (see also [2]).

An attempt had been made to determine the response of a cylindrical shell to an acoustic wave. The form of the functions dealt with in the analysis made it difficult to obtain accurate explicit results. If the obstacle is taken to be spherical in shape, we still have a practical though highly simplified model of an actual physical structure; moreover, the problem becomes mathematically simpler, admitting of exact solutions for the deformation and accompanying strains in the elastic body. It was therefore decided to treat the case of the sphere in detail.

2. Forced vibrations of a thin spherical shell. Consider a closed shell of thickness h with $h \ll R$ where R is the radius of the middle surface. The motion of any closed oval[2] shell, in particular a spherical shell is, by a theorem of Jellett [3], primarily extensional. Therefore, the general membrane theory of shells is applicable. If we locate the origin of our coordinate system at the center of the shell and choose as the z-axis the direction of propagation of the incoming wave, then we have the additional simplification of symmetrical loading (see Fig. 1A).

The equations of dynamic equilibrium for an element of shell may, therefore, be written [4],

$$\frac{\partial}{\partial \theta}(N_\theta \sin \theta) - N_\phi \cos \theta - \rho h \frac{\partial^2 v}{\partial t^2} R \sin \theta = 0, \tag{2.1}$$

$$N_\theta + N_\phi - \left(s - \rho h \frac{\partial^2 w}{\partial t^2}\right) R = 0. \tag{2.2}$$

Here N_ϕ and N_θ are the normal forces/unit length acting on the sides of the element (see Fig. 1B), s is the external load (radial in direction), ρ is the shell density, and v and w are the tangential and radial components of the displacement.

N_ϕ and N_θ can be eliminated from the Eqs. (2.1) and (2.2) by the use of Hooke's

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[1]Numbers in brackets refer to the Bibliography at the end of the paper.

[2]Principal radii of curvature finite and of the same sign.

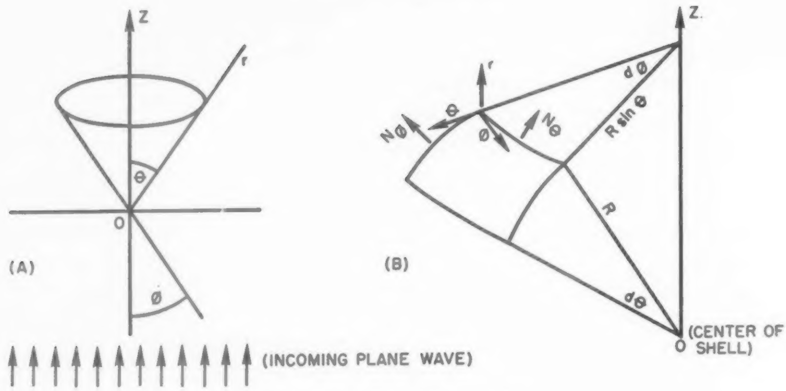

FIG. 1. Geometry of the problem.

law and the expressions for the strains in spherical coordinates. The resulting equations can be further simplified by the introduction of the dimensionless variables (E is Young's modulus),

$$w^* \equiv w/R; \qquad v^* \equiv v/R; \qquad \tau^2 \equiv t^2/t_0^2 \equiv Et^2/(R^2\rho); \qquad s^* \equiv s/s_0 \equiv Rs/(Eh).$$

After some manipulation, we obtain for the equations of motion the following relations in w^* and v^* only (ν is the Poisson's ratio).

$$\begin{aligned} M(v^*) = {} & 2(1-\nu^2)\frac{\partial^2 v^*}{\partial\tau^2} + (1-\nu^2)(1-\nu)\frac{\partial^4 v^*}{\partial\tau^4} + 2\nu v^* + \nu(1-\nu)\frac{\partial^2 v^*}{\partial\tau^2} \\ & - 2\frac{\partial^2 v^*}{\partial\theta^2} - (1-\nu)\frac{\partial^4 v^*}{\partial\tau^2\,\partial\theta^2} + 2v^*\cot^2\theta + (1-\nu)(\cot^2\theta)\frac{\partial^2 v^*}{\partial\tau^2} \\ & - 2(\cot\theta)\frac{\partial v^*}{\partial\theta} - (1-\nu)(\cot\theta)\frac{\partial^3 v^*}{\partial\theta\,\partial\tau^2} + (1+\nu)\left[(\cot\theta)\frac{\partial v^*}{\partial\theta}\right. \\ & \left. - v^*\csc^2\theta + \frac{\partial^2 v^*}{\partial\theta^2}\right] = (1-\nu^2)\frac{\partial s^*}{\partial\theta}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} L(w^*) = {} & (\cot\theta)\left(\frac{\partial w^*}{\partial\theta} + \frac{\partial^3 w^*}{\partial\tau^2\,\partial\theta}\right) + 2w^* + \frac{\partial^2 w^*}{\partial\theta^2} + \frac{\partial^4 w^*}{\partial\tau^2\,\partial\theta^2} \\ & - (1-\nu^2)\frac{\partial^4 w^*}{\partial\tau^4} + (1-\nu)\frac{\partial^2 w^*}{\partial\tau^2} - 2(1+\nu)\frac{\partial^2 w^*}{\partial\tau^2} \\ = {} & \cot\theta\,\frac{\partial s^*}{\partial\theta} - (1-\nu^2)\frac{\partial^2 s^*}{\partial\tau^2} + \frac{\partial^2 s^*}{\partial\theta^2} + (1-\nu)s^*. \end{aligned} \tag{2.4}$$

3. Acoustic wave propagation. The linearized theory of wave propagation yields the acoustic wave equation

$$\Delta\phi - \lambda^2\phi_{,\tau\tau} = 0, \tag{3.1}$$

where ϕ is the velocity potential, $\Delta\phi$ is the Laplacian in spherical coordinates, $\lambda^2 =$

$R^2/(t_0^2c^2) = E/(\rho c^2)$, and c is the acoustic speed of the fluid. The pressure perturbation P is given by

$$P = -p^*\phi_{,\tau}(r, \theta, \tau),$$
$$p^* = \rho_f R^2/t_0^2 = \rho_f E/\rho, \tag{3.2}$$

where ρ_f is the fluid density. The applied stress, s, of (2.2) must, of course, be the same as the acoustic pressure P at the surface of the sphere. We have, in fact

$$s^* = -P/s_0 = (p^*/s_0)\phi_{,\tau}(1, \theta, \tau) = (\rho_f R/\rho h)\phi_{,\tau}(1, \theta, \tau)$$
$$\equiv \beta\phi_{,\tau}(1, \theta, \tau).$$

4. The interaction problem. We now pose the following problem. An incoming plane wave, ϕ_0, obeying (3.1) impinges on an elastic spherical shell causing it to vibrate according to (2.3) and (2.4). The vibrating body acts as a scatterer and, to a lesser extent, as a radiator. The outgoing waves (scattered and radiated) also obeying (3.1), in turn influence the nature of the vibrations. At the surface of the shell, the radial velocity of the fluid, $\phi, r(1, \theta, \tau)$ must be equal to the radial velocity of the shell, $w^*_{,\tau}(\theta, \tau)$. We wish to determine the motion of the sphere and the pressure distribution associated with the incoming and outgoing waves.

The pressure associated with the incident wave is taken to be [1]

$$P = \begin{cases} Q_0 \exp[\delta(z - \tau/\lambda)] & z \leq \tau/\lambda \\ 0 & z > \tau/\lambda \end{cases} \tag{4.1}$$

so that the initial velocity potential is

$$\phi_0 = \begin{cases} (Q_0\lambda/p^*\delta) \exp[\delta(z - \tau/\lambda)] - (Q_0\lambda/p^*\delta) & z \leq \tau/\lambda \\ 0 & z > \tau/\lambda. \end{cases} \tag{4.2}$$

For simplicity let ψ_0, ψ, χ, W and V be defined as:

$$\phi_0 \equiv (Q_0/p^*)\psi_0, \qquad \phi \equiv (Q_0/p^*)(\psi_0 + \psi) \equiv (Q_0/p^*)\chi,$$
$$w^* \equiv (Q_0/p^*)W, \qquad v^* \equiv (Q_0/p^*)V. \tag{4.3}$$

Then our boundary value problem is defined by the following set of equations[3].

$$L(W) = \beta[\chi_{,\theta\tau}(1, \theta, \tau) \cot\theta - (1 - \nu^2)\chi_{,\tau\tau\tau}(1, \theta, \tau)$$
$$+ \chi_{,\theta\theta\tau}(1, \theta, \tau) + (1 - \nu)\chi_{,\tau}(1, \theta, \tau), \tag{4.4}$$

$$M(V) = \beta(1 - \nu^2)\chi_{,\theta\tau}(1, \theta, \tau), \tag{4.5}$$

$$\Delta\psi - \lambda^2\psi_{,\tau\tau} = 0, \tag{4.6}$$

$$W_{,\tau}(\theta, \tau) = \chi_{,r}(1, \theta, \tau), \qquad \text{[boundary condition]} \tag{4.7}$$

$$\psi_0 = \begin{cases} (\lambda/\delta) \exp[\delta(z - \tau/\lambda)] - \lambda/\delta & z \leq \tau/\lambda \\ 0 & z > \tau/\lambda. \end{cases} \tag{4.8}$$

---

[3]The operators L and M which appear here are those given in (2.3) and (2.4).

These equations will be more easily handled if Fourier transforms are first introduced to eliminate the time dependence. Denote the transform of a function F by F^T. Then F and F^T are related by

$$F^T(r, \theta, \xi) = \int_{-\infty}^{\infty} F(r, \theta, \tau) \exp(-i\xi\tau)\, d\tau, \tag{4.9a}$$

$$Im\, \xi = -i\alpha, \qquad \alpha > 0,$$

$$F(r, \theta, \tau) = \frac{1}{2\pi} \int_{-i\alpha+\infty}^{-i\alpha-\infty} F^T(r, \theta, \xi) \exp(i\xi\tau)\, d\xi. \tag{4.9b}$$

Applying (4.9a) to (4.4) through (4.8) we obtain

$$L^T(W^T) = i\beta\xi[\chi^T_{.\theta} \cot \theta + \chi^T_{.\theta\theta}(1, \theta, \xi) + (1 - \nu)\chi^T + \xi^2(1 - \nu^2)\chi^T], \tag{4.10}$$

where

$$L^T = (\cot \theta)\left(\frac{\partial}{\partial \theta} - \xi^2 \frac{\partial}{\partial \theta}\right) + 2 + \frac{\partial^2}{\partial \theta^2} - \xi^2 \frac{\partial^2}{\partial \theta^2} - (1 - \nu^2)\xi^4$$

$$- (1 - \nu)\xi^2 + 2(1 + \nu)\xi^2,$$

$$M^T(V^T) = i\beta\xi(1 - \nu^2)\chi^T_{.\theta}\,, \tag{4.11}$$

where

$$M^T = -2(1 - \nu^2)\xi^2 + (1 - \nu^2)(1 - \nu)\xi^4 + 2\nu - \nu(1 - \nu)\xi^2$$

$$- 2\frac{\partial^2}{\partial \theta^2} + (1 - \nu)\xi^2 + 2 \cot^2 \theta - (1 - \nu)\xi^2 \cot^2 \theta$$

$$- 2(\cot \theta)\frac{\partial}{\partial \theta} + (1 - \nu)\xi^2(\cot \theta)\frac{\partial}{\partial \theta} + (1 + \nu)\left[(\cot \theta)\frac{\partial}{\partial \theta} - \csc^2 \theta + \frac{\partial^2}{\partial \theta^2}\right]$$

$$\Delta\psi^T + \lambda^2\xi^2\psi^T = 0, \tag{4.12}$$

$$i\xi W^T(\theta, \xi) = \chi^T_{.r}(1, \theta, \xi), \tag{4.13}$$

$$\psi_0^T = -\frac{\exp(-iz\lambda\xi)}{i\xi[(\delta/\lambda) + i\xi]} = -f(\xi) \exp(-i\lambda r\xi \cos \theta);$$

$$f(\xi) \equiv \{i\xi[(\delta/\lambda) + i\xi]\}^{-1}. \tag{4.14}$$

It can be shown, using the method of separation of variables on (4.12), that ψ^T is of the form

$$r^{-1/2}[J_{n+1/2}(\lambda\xi r) + CY_{n+1/2}(\lambda\xi r)]P_n(\cos \theta).$$

Since the waves associated with ψ^T must be outwardly moving, take $C = -i$ and write:

$$\psi^T = \sum_{n=0}^{\infty} A_n^T(\xi) h_n^{(2)}(\lambda\xi r) P_n(\cos \theta), \tag{4.15}$$

where

$$h_n^{(2)}(\lambda\xi r) = (\pi/2\lambda\xi r)^{1/2}[J_{n+1/2}(\lambda\xi r) - iY_{n+1/2}(\lambda\xi r)].$$

ψ_0^T may be expanded similarly

$$\begin{aligned}\psi_0^T(r, \theta, \xi) &= -f(\xi) \sum_{n=0}^{\infty} (2n + 1)(-i)^n j_n(\lambda\xi r)P_n(\cos \theta),\\ &\equiv \sum_{n=0}^{\infty} B_n(\xi)j_n(\lambda\xi r)P_n(\cos \theta),\end{aligned} \tag{4.16}$$

where

$$j_n(\lambda\xi r) = (\pi/2\lambda\xi r)^{\frac{1}{2}}J_{n+\frac{1}{2}}(\lambda\xi r).$$

From the work of Lamb [5], we know that the complete solutions for W^T and V^T may be written:

$$W^T(\theta, \xi) = \sum_{n=0}^{\infty} W_n^T(\xi)P_n(\cos \theta), \tag{4.17}$$

$$V^T(\theta, \xi) = -\sum_{n=0}^{\infty} V_n^T(\xi)P_n^1(\cos \theta) = -\sum_{n=0}^{\infty} V_n^T P_n'(\cos \theta) \sin \theta. \tag{4.18}$$

Substituting these expressions into (4.10), (4.11) and (4.13), we get three algebraic equations for the three unknowns W_n^T, V_n^T and A_n^T, which are readily solved to give:

$$W_n^T = \frac{(2n + 1)(-i)^n C_2}{\lambda\xi(i\xi)^2[(\delta/\lambda) + i\xi][C_1\lambda h_n^{(2)\prime} + iC_2 h_n^{(2)}]}, \tag{4.19}$$

$$V_n^T = \frac{-(2n + 1)(-i)^n(1 + \nu)\beta}{\lambda\xi(i\xi)[(\delta/\lambda) + i\xi][C_1\lambda h_n^{(2)\prime} + iC_2 h_n^{(2)}]}, \tag{4.20}$$

$$A_n^T = -\tfrac{1}{2}\left[1 + \frac{C_1\lambda h_n^{(1)\prime} + iC_1 h_n^{(1)}}{C_1\lambda h_n^{(2)\prime} + iC_2 h_n^{(2)}}\right]\frac{1}{\xi^2}(2n + 1)(-i)^n, \tag{4.21}$$

where

$$\begin{aligned}C_1 &= (1 - \xi^2)(n^2 + n) - 2 + (1 - \nu^2)\xi^4 + (1 - \nu)\xi^2 - 2(1 + \nu)\xi^2,\\ C_2 &= i\beta\xi[(1 - \nu^2)\xi^2 - (n^2 + n) + (1 - \nu)].\end{aligned}$$

The quantities of physical interest are the stresses, radial acceleration and the total pressure distribution. These can now all be found, at least in principle, from (4.19), (4.20), (4.21) and the inversion formula (4.9b).

5. Numerical results for the shell. For each vibrational mode, the transforms of the stress components will be linear combinations of W_n^T and V_n^T; e.g. for the nth mode, N_ϕ^T will be given by

$$[Q_0R/(1 - \nu^2)\beta]\left[(1 + \nu)W_n^T P_n(\cos \theta) - V_n^T P_n^1(\cos \theta) \cot \theta - \nu V_n^T \frac{\partial P_n^1(\cos \theta)}{\partial \theta}\right].$$

The transform of the total radial acceleration, $\partial^2 w/\partial\tau^2$, will be $-(Q_0R^2/Eh\beta)\xi^2 W^T$ or, for the nth mode only

$$-(Q_0R^2/Eh\beta)\xi^2 W_n^T P_n(\cos \theta).$$

From (4.19) and (4.20) we see that these expressions are regular in ξ except for a finite number of poles. The theorem of residues and Jordan's lemma can therefore be used to evaluate the integral of (4.9b). The computation was carried out in detail for the three

lowest modes, and the results are shown for the point $\theta = \pi$ in Figs. 2 and 3. N_ϕ only is shown in Fig. 2 since the two tangential stress components are equal for $\theta = \pi$.

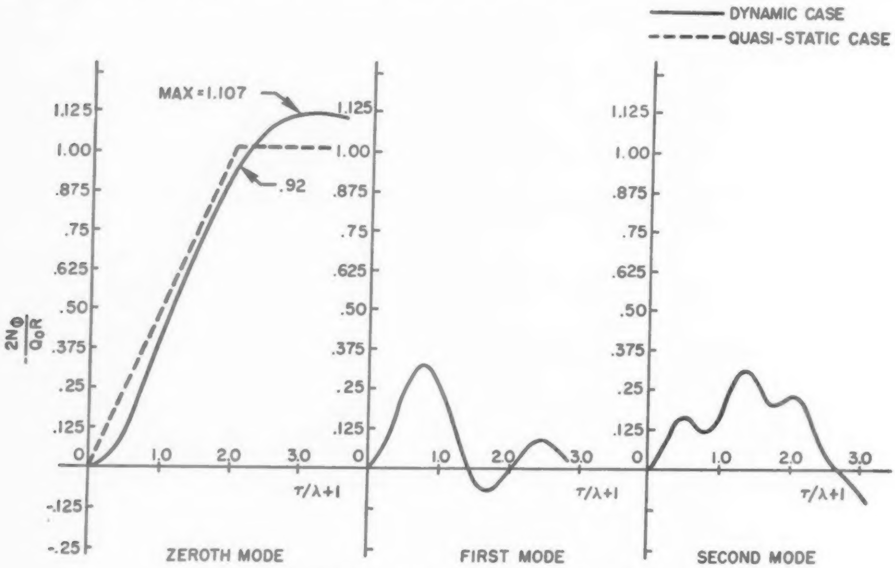

FIG. 2. Hoop stress at $\theta = \pi$ for the first 3 modes.

The parameters were taken as

$$\beta = 17, \qquad \delta = 0, \qquad \lambda^2 = 25/3, \qquad \nu = .3,$$

the values appropriate to a step wave of infinite length impinging on a steel shell in water.

From Fig. 2 it becomes evident that the series representing the stresses in the shell converge rapidly enough so that a very good approximation to the total stress may be obtained by considering just the first few modes. Of these, the zeroth mode, which has the steady state solution as its limiting form, is the most important. In this connection it should be noted that the curves in Fig. 2, which were made for $\theta = \pi$, show the largest stresses which can occur in the first and second modes at any time. The zeroth mode is, of course, θ — independent.

The computation of the acceleration is not so simple. A picture of the total acceleration cannot be gained by looking at the lower modes. In fact, it would seem from Fig. 3 that a few terms of the series $\sum_{n=0}^{\infty} \partial^2 w_n/\partial\tau^2\ P_n\ (\cos\theta)$ will not approximate with acceptable accuracy the total acceleration for all τ and θ. The rate of change of radial acceleration is very great at $\tau/\lambda = -1$, $\theta = \pi$; probably the total radial acceleration will be discontinuous at $\tau/\lambda = -1$, $\theta = \pi$. That this is actually so can be seen by recalling that the incoming wave reaches the point $(1, \theta)$ on the sphere at time $\tau/\lambda = \cos\theta$. In particular, the point of initial impact, $\theta = \pi$, is reached at $\tau/\lambda = -1$. This means that the pressure and, therefore, the radial acceleration as well are discontinuous and that the series representing them do not converge uniformly for $\tau/\lambda = \cos\theta$. The solutions

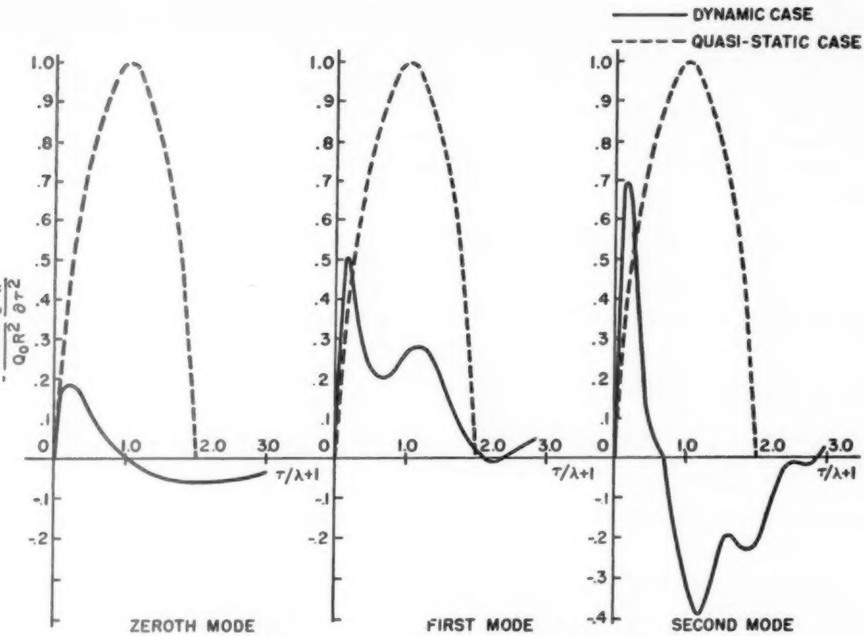

FIG. 3. Radial acceleration at $\theta = \pi$ for the first 3 modes.

for these points must be found in closed form, i.e. when the transform, $\xi^2 W^T$ (4.17), is inverted, the order of summation and integration cannot be interchanged. This difficulty, associated with the use of the series expansion as a method of solution, will be encountered again in Sec. 6 when the resultant pressure distribution is discussed. By anticipating the results of that section, we can find the initial value of $\partial^2 w/\partial \tau^2$ at $\theta = \pi$.

Equation (2.2) gives, for $\tau/\lambda = -1$

$$s = \rho h \frac{\partial^2 w}{\partial t^2} = \frac{Eh}{R^2}\frac{\partial^2 w}{\partial \tau^2},$$

$$\frac{-4s}{Q_0} = \frac{-4Eh}{Q_0R^2}\frac{\partial^2 w}{\partial \tau^2}.$$

Taking

$$s = -2Q_0[(6.6),\ \theta = \pi]$$

we have

$$\frac{-4Eh}{Q_0R^2}\frac{\partial^2 w}{\partial \tau^2} = 8.$$

Thus, while we can obtain a very good approximation to the stresses by considering only the first few modes or even the lowest mode by itself, the same is definitely not true of the acceleration. The actual initial acceleration is about 40 times as great as the maximum acceleration in the zeroth mode. The agreement obtained by considering the

first and second modes along with the zeroth is not appreciably better; the results still differ by more than a factor of 6.

Quasi-Static Case. It is of interest to compare the hoop stresses for the zeroth mode with those for the quasi-static case; i.e. for the case in which both scattering and the inertia forces due to deformation are neglected.

Since the effects of only the incident wave are considered, the pressure at time t is given by

$$\text{pressure} = \begin{cases} \dfrac{Q_0 2\pi R(tc + R)}{4\pi R^2} = \dfrac{Q_0[(\tau/\lambda) + 1]}{2} & -1 \le \tau/\lambda \le 1 \\ Q_0 & 1 \le \tau/\lambda. \end{cases}$$

The stresses are given by

$$N_\theta = N_\phi = \begin{cases} \dfrac{-RQ_0[(\tau/\lambda) + 1]}{4} & -1 \le \tau/\lambda \le 1 \\ -RQ_0/2 & 1 \le \tau/\lambda. \end{cases}$$

$-2N_\phi/(RQ_0)$ appears as the dotted line in Fig. 2, where it can be seen that the very simple quasi-static case approximates the more exact dynamic case very closely. The stress developments, while slightly out of phase, are essentially parallel with a difference in maxima of only 10 per cent.

For $-1 \le \tau/\lambda < 1$, there will be an unbalanced force, due to the incident wave, acting on the sphere. This will result in a rigid-body acceleration in the z direction. The magnitude of the force is

$$Q_0\pi R^2[1 - (\tau/\lambda)^2]$$

therefore

$$4\pi R^2 h\rho \frac{\partial^2 z}{\partial \tau^2} = Q_0\pi R^2[1 - (\tau/\lambda)^2]$$

or

$$\frac{4Eh}{Q_0 R^2}\frac{\partial^2 z}{\partial \tau^2} = [1 - (\tau/\lambda)^2].$$

This has been plotted for purposes of comparison in Fig. 3.

The formal resemblance is greatest for the first mode. The maximum acceleration in the zeroth mode is only one fifth the rigid-body acceleration. For higher modes, the maxima move closer to the rigid-body value of 1. By referring to the previous section, however, we see that the maximum total acceleration associated with deformation is 8 times that of the rigid body.

6. Resultant pressure distribution. From (3.2), (4.1), (4.3), (4.15), and (4.21) and (4.9b) the total pressure is known to be

$$P_{\text{Total}} = P_{\text{I}}\,(\text{incident}) + P_{\text{II}} = -p^*\phi_{,\tau} = -Q_0\chi_{,\tau}\,,$$

$$= Q_0 - \frac{Q_{0i}}{2\pi}\int_{-i\alpha-\infty}^{-i\alpha+\infty} \exp(i\xi\tau)\psi^T\xi\, d\xi,$$

$$= Q_0 + \frac{Q_{0i}}{2\pi}\int_{-i\alpha-\infty}^{-i\alpha+\infty} \xi \exp(i\xi\tau) \sum_{n=0}^{\infty} \frac{(-i)^n(2n+1)}{2\xi^2}\, h_n^{(2)}(\lambda\xi r)\Bigg[1$$

$$+ \frac{C_1\lambda h_n^{(1)\prime} + iC_2h_n^{(1)}}{C_1\lambda h_n^{(2)\prime} + iC_2h_n^{(2)}}\Bigg]P_n(\cos\theta)\, d\xi. \tag{6.1}$$

Unfortunately, the order of summation and integration in (6.1) cannot be interchanged for all θ, τ and r; the total pressure cannot be approximated by the sum of the first n vibrational modes. This can be seen for the specific case $r = 1$, $\tau = -\lambda$, $\theta = \pi$ as follows. At the moment of initial impact, $\tau = -\lambda$, we have for each mode and for all θ, $N_\theta = N_\phi = \partial^2 v_n/\partial\tau^2 = \partial^2 w_n/\partial\tau^2 = 0$. The equations of equilibrium, (2.1) and (2.2), require that the pressure at the surface of the sphere must likewise vanish for each mode at $\tau = -\lambda$, so that the total pressure would be zero for all θ. We should expect, however, from what is known of the theory of scattering of plane waves, that the pressure Q_0 would be doubled and not reduced to zero.

ψ^T must be found in closed form if P_{Total} is to be evaluated. It has not proved feasible to carry out the summation for all θ, r and τ. However, it was noted that the expansions for ψ_0^T and ψ^T are very similar for $r = 1$, and ξ very large or $[(\tau/\lambda) - \cos\theta]$ very small. This fact can be used to obtain the pressure at the surface of the sphere for $\tau/\lambda \approx \cos\theta$.

By referring to the expression for $\psi_0^T(4.16)$, it is seen that if we set $\lambda\xi = \zeta$, the pressure associated with the incident wave may be written

$$P_1 = Q_0 = -\frac{Q_{0i}}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{\exp(i\zeta\tau/\lambda)}{\zeta} \sum_{n=0}^{\infty} j_n(\zeta r)P_n(\cos\theta)(2n+1)(-i)^n\, d\zeta.$$

For $\zeta \gg \beta$ and $r = 1$, P_{II}, (see 6.1), becomes, to first order

$$P_{\text{II}} = \frac{Q_0}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{\exp(i\zeta\tau/\lambda)}{\zeta} \sum_{n=0}^{\infty}$$

$$\cdot\, \frac{\sin[\zeta - (n\pi/2) - \pi/2]}{\zeta}(2n+1)(-i)^nP_n(\cos\theta)\, d\zeta \tag{6.2}$$

and P_I reduces to

$$P_{\text{I}} = -\frac{Q_{0i}}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{\exp(i\zeta\tau/\lambda)}{\zeta} \sum_{n=0}^{\infty}$$

$$\cdot\, \frac{\cos[\zeta - (n\pi/2) - \pi/2]}{\zeta}(2n+1)(-i)^nP_n(\cos\theta)\, d\zeta. \tag{6.3}$$

Equations (6.3) and (4.14) tell us that

$$\zeta \exp(-i\zeta\cos\theta) \underset{\zeta\to\infty}{\longrightarrow} \sum_{n=0}^{\infty} \cos[\zeta - (n\pi/2) - \pi/2](2n+1)(-i)^nP_n(\cos\theta)$$

so that to first order:

$$P_{II} = \frac{Q_0}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{\exp\{i\zeta[(\tau/\lambda) - \cos\theta]\}}{\zeta} \exp[i(\pi/2)\cos\theta]\, d\zeta,$$

$$\begin{aligned} &= Q_0 \exp[i(\pi/2)(\cos\theta + 1)] && \text{for } 0 \leq [(\tau/\lambda) - \cos\theta] \ll .05, \\ &= 0 && \text{for } [(\tau/\lambda) - \cos\theta] < 0. \end{aligned} \tag{6.4}$$

Higher order terms may be obtained in the same way. To second order:

$$P_{II} = \frac{Q_0}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{\exp\{i\zeta[(\tau/\lambda) - \cos\theta]\}}{\zeta}\left\{\exp[i(\pi/2)\cos\theta] + \left[\frac{\beta+1}{\zeta}\right][1 + i\exp(i(\pi/2)\cos\theta)] - \frac{\pi}{2\zeta}\exp[i(\pi/2)\cos\theta]\right\} d\zeta,$$

$$\begin{aligned} &= Q_0\{\exp[i(\pi/2)\cos\theta + i\pi/2] - [(\tau/\lambda) - \cos\theta][1+\beta][1 \\ &\quad + \exp(i(\pi/2)\cos\theta + i\pi/2)] + \pi/2[(\tau/\lambda) - \cos\theta][\exp(i(\pi/2)\cos\theta)]\} \\ &\qquad\qquad \text{for } 0 \leq [(\tau/\lambda) - \cos\theta] \ll .05, \\ &= 0 \qquad\qquad \text{for } [(\tau/\lambda) - \cos\theta] < 0. \end{aligned} \tag{6.5}$$

Additional terms will be of little value since the representation is valid only for $\zeta \gg \beta$, $[(\tau/\lambda) - \cos\theta] \ll .05$. The terms found so far, however, are sufficient to tell us some things of importance.

The incoming wave will reach the point $(1, \theta)$ on the sphere at time $\tau/\lambda = \cos\theta$. The pressure on impact for each θ is given by[4]

$$P_{\text{Total}} = P_I + Re\, P_{II} = Q_0 + Q_0 \cos[(\pi/2)(\cos\theta + 1)]. \tag{6.6}$$

At $\theta = \pi$, the outermost point of the sphere, the first effect is that of a plane wave hitting a rigid wall and we have

$$P_{II} = P_I\,; \qquad P_{\text{Total}} = 2Q_0\,.$$

At $\theta = \pi/2$, the wave just grazes the sphere and therefore

$$P_{II} = 0; \qquad P_{\text{Total}} = Q_0\,.$$

As θ varies from π to $\pi/2$, the initial pressure varies continuously from $2Q_0$ to Q_0. If the sphere were rigid, the steady-state pressure distribution would be given by

$$P_{\text{Total}} = P_I \qquad 0 \leq \theta \leq \pi.$$

The results of Sec. 5 indicate that we will have asymptotic values $RQ_0/2$ for the stresses and zero for the radial acceleration, which also correspond to a uniform pressure of $Q_0 = P_I$.

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[4]The elastic waves in the shell will travel more rapidly than the acoustic wave and will result in a pressure, $P \neq 0$, at $(1, \theta)$ before the time $\tau/\lambda = \cos\theta$, $\theta \neq \pi$. However, this effect is negligibly small compared with the one we are considering.

At present, this is about all that can be said on the subject of the pressure distribution. While the series method is very convenient for finding the stresses and may be adapted to give results for the radial acceleration, it does not lend itself to an analysis of the pressure distribution. It would seem that, for this aspect of the problem, another approach is called for.

7. Remarks. In a paper which appeared after this work was completed, J. H. Huth and J. D. Cole [6] consider the related problem of the effect of a shock wave on an airborne elastic spherical shell.

This problem is more difficult than the one treated here, and the authors have introduced the simplifying assumption that the effect of the diffracted wave on the shell may be neglected, or, in other words, that the effect on the applied pressure of both the motion of the sphere and the sphere itself may be neglected.

The stresses, but not the radial acceleration, are computed. The problem of finding the resultant pressure distribution does not, of course, arise.

It would be of interest to compare their results for the limiting case of a very weak shock (i.e. speed of shock wave $\rightarrow$ speed of sound) with the stresses that would be obtained here by using for β and λ (see Sec. 3) the values appropriate to a steel shell in air. Such computations are now under way.

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—NOTES—

ON THE CONDITIONS OF VALIDITY OF RIEMANN'S METHOD OF INTEGRATION*

By AUREL WINTNER (*The Johns Hopkins University*)

1. Introduction. The traditional treatment of Riemann's classical formula depends on certain additional assumptions which are not necessary for the existence of a solution (Sec. 2) but, being based on the adjoint equation (Sec. 3), are necessary for the customary introduction of Riemann's Green function.

The purpose of this note is to point out the resulting complication and to show that it can be overcome by the application of a simple device in the proof. The main result is that italicized in Sec. 8.

It will be sufficient to consider only one of the standard cases, the case in which the boundary values of the unknown function itself are assigned along a path consisting of two characteristics which meet at a point. In fact, it will be clear from the nature of the arguments to be applied that all considerations remain valid for the case in which the data are, for instance, Cauchy data proper, the case in which the unknown function and its normal derivative are assigned along a continuously differentiable path on which no direction is a characteristic direction.

2. Picard's theorem. On the closed rectangle $0 \leqq x \leqq \xi$, $0 \leqq y \leqq \eta$, which will be denoted by $Q = Q(\xi, \eta)$, consider Riemann's hyperbolic partial differential equation

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0, \tag{1}$$

with the boundary data

$$u(x, \eta) = \varphi(x), \qquad u(\xi, y) = \psi(y), \tag{2}$$

where $\varphi(x)$ $(0 \leqq x \leqq \xi)$ and $\psi(y)$ $(0 \leqq y \leqq \eta)$ are given functions satisfying

$$\varphi(\xi) = \psi(\eta). \tag{3}$$

The appropriate conditions, to be imposed on (a, b, c) and (φ, ψ), respectively, are as follows: (i) the coefficient functions $a(x, y)$, $b(x, y)$, $c(x, y)$ are continuous on Q and (ii) the boundary functions $\varphi(x)$, $\psi(y)$ are continuously differentiable. In fact, the situation is as follows.

The pair conditions (i)-(ii) assures that there exists on the rectangle Q a unique function $u = u(x, y)$ having the following properties:

$$u_x\ ,\ u_y \text{ and } u_{xy}(= u_{yx}) \text{ exist and are continuous,} \tag{4}$$

the relation (1) is an identity on Q and the two equations (2) are identities on the sides $y = \eta$, $x = \xi$ of Q.

This will be referred to as Picard's theorem. It can be proved by the method of

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successive approximations and is the precise formulation of what is actually proved in Picard's writings on the subject.[1]

3. The formal adjoint. In contrast to what is supplied by Picard's theorem, Riemann's method of integration replaces Eq. (1) by its adjoint,

$$v_{xy} - [a(x, y)v]_x - [b(x, y)v]_y + c(x, y)v = 0, \tag{1 bis}$$

to which certain boundary data

$$v(x, \eta) = \rho(x), \qquad v(\xi, y) = \sigma(y) \tag{2 bis}$$

satisfying

$$\rho(\xi) = \sigma(\eta) \tag{3 bis}$$

are assigned.

The traditional presentation of Riemann's method[2] depends on an application of Picard's theorem to the case in which the Eq. (1) and the boundary condition (2) are replaced by the adjoint equation (1 *bis*) and a certain boundary condition (2 *bis*), respectively. A moment's reflection shows, however, that this application is inadmissible under the assumptions (i)-(ii) placed on (1)-(2) by Picard's theorem. For, since (i) merely assumes the continuity of $a(x, y)$ and $b(x, y)$, the derivatives $a_x(x, y)$, $b_y(x, y)$ need not exist (or, if they do, they need not be continuous) and so Eq. (1 *bis*) cannot in general be expanded into

$$v_{xy} + A(x, y)v_x + B(x, y)v_y + C(x, y)v = 0$$

with certain functions A, B, C, still less with continuous functions A, B, C, as it is required by (i) when Eq. (1 *bis*) is identified with Eq. (1).

4. Methodical remarks. The purpose of this note is to point out an easy way out of the implications of this predicament.

For the case of the elliptic analogues of the hyperbolic equation (1), a similar difficulty was circumvented by Lichtenstein,[3] by using an "integrated form" of the adjoint. In the hyperbolic case at hand, a simple and direct approach, which makes explicit a function space allowable for the adjoint under the original assumption (i) on Eq. (1), can be obtained.

Today, Lichtenstein's result can be interpreted as a manifestation of L. Schwartz's distribution theory[4]. From the point of view of the theory of distributions, the success of the explicit approach in the hyperbolic case will center around the following formal fact. If the set-function $I^q(f)$ (of the set q) is the integral of a continuous $f = f(x, y)$ over a rectangle $q = (x_1 \leqq x \leqq x_2\ , y_1 \leqq y \leqq y_2)$, and if f is one of the (continuous)

[1]See, e.g., E. Picard, *Leçons sur quelques types simples d'équations aux dérivées partielles avec des applications à la physique mathématique*, 1927, pp. 123-134. For an extension to the case of certain non-linear equations, a case the treatment of which is not based on the method of successive approximations but on that of equicontinuous functions, see P. Hartman and A. Wintner, Amer. J. of Math. **74**, 836-843 (1952)

[2]See, e.g. E. Picard, *op. cit.*, pp. 147-151 or G. Darboux, *Leçons sur la théorie générale des surfaces*, 2, 71-81 (1889). The literature consulted on the extension of the classical conditions on Riemann's method (see, e.g., G. S. S. Ludford, J. of Ratl. Mech. and Anal. **3**, 77-88 (1954), where further references are given) does not go into the problems of the adjoint considered in this paper

[3]L. Lichtenstein, Bull. Acad. Polon. des Sciences **1931**, 571-598

[4]L. Schwartz, *Théorie des distributions*, vol. 1-2, 1950-1951

derivatives z_x, z_y, z_{xy} of a function $z = z(x, y)$, then the evaluation of $I^a(f)$ involves no differentiation of $z(x, y)$ in any of the three cases $f = z_x, z_y, z_{xy}$ (it involves integrations in the first two of the three cases).

5. Reformulation of the adjoint problem. Suppose that the coefficient functions a, b, c of Eq. (1) are subject only to the continuity assumption (i) on Q. Then Eq. (1 *bis*), as it stands, is meaningless in general. But it appears in a meaningful form if it is integrated over the rectangle having (x, y) and (ξ, η) as opposite vertices, where (x, y) is any point of the given rectangle $Q = (0 \leqq x \leqq \xi, 0 \leqq y \leqq \eta)$. In fact, the formal result of this integration is

$$v(\xi, \eta) - v(\xi, y) - v(x, \eta) + v(x, y)$$
$$- \int_y^\eta a(s, t)v(s, t) \Big|_{s=x}^{s=\xi} dt - \int_x^\xi b(s, t)v(s, t) \Big|_{t=y}^{t=\eta} ds + \int_x^\xi \int_y^\eta c(s, t)v(s, t)\, ds\, dt = 0.$$

In view of the boundary data (2 *bis*) and the condition (3 *bis*), this can be written in the form

$$v(x, y) = -\rho(\xi) + \rho(x) + \sigma(y) - \int_x^\xi \int_y^\eta c(s, t)v(s, t)\, ds\, dt$$
$$+ \int_y^\eta \{a(\xi, t)v(\xi, t) - a(x, t)v(x, t)\}\, dt + \int_x^\xi \{b(s, \eta)v(s, \eta) - b(s, y)v(s, y)\}\, ds,$$

[where $-\rho(\xi)$ can be replaced by $-\sigma(\eta)$]. Let the latter formula for $v(x, y)$, a formula which is an integral equation for the unknown v, with a, b, c and ρ, σ as data, be referred to as Eq. (1*).

Under appropriate assumptions of differentiability, Eq. (1*) is equivalent to Eq. (1 *bis*) and (2 *bis*) together. But Eq. (1*) is meaningful under the following pair of assumptions also: (i*) the continuity assumption (i) of Sec. 2 for $a(x, y)$, $b(x, y)$ $c(x, y)$ on Q and (ii*) the continuity of $\rho(x)$, $\sigma(y)$ on the respective intervals $0 \leqq x \leqq \xi$, $0 \leqq y \leqq \eta$ [note that (ii*) requires of ρ, σ less than (ii) in Sec. 2 requires of φ, ψ].

6. The dual of Picard's theorem. Corresponding to the circumstance that the formulation (1*) of the adjoint problem is free of any differentiation, it is natural to extend the solution class, defined by condition (4) in Picard's theorem, as follows: a "solution" $v = v(x, y)$ should mean any function defined on Q in such a way that

$$v(x, y) \text{ is continuous} \tag{4*}$$

on Q and satisfies the Eq. (1*) as an identity on Q. The fundamental fact, representing the true dual of Picard's theorem, can then be formulated as the following theorem.

If (1) and (2) are replaced by (1*) and, at the same time, (i), (ii) and (4) are replaced by (i*), (ii*) and (4*), respectively [which means that (i) is retained but both (ii) and (4) are extended to the assumption of mere continuity], then there exists on Q exactly one solution $v(x, y)$.

The proof of this dual of Picard's theorem depends on successive approximations. To this end, let $v_0(x, y) \equiv 0$ on Q and, if $v_n(x, y)$ has already been defined on Q as a continuous function, let $v_{n+1}(x, y)$ denote the function which results if v is replaced by v_n in the sum which represents the expression on the right of Eq. (4*). It then follows that $v_1(x, y)$, $v_2(x, y)$, $\cdots$ is a sequence of continuous functions which tend to a limit function uniformly on Q and that, if $v(x, y)$ denotes the limit function, then $v(x, y)$ is a

solution of Eq. (4*). The details are of a routine nature and will therefore be omitted. The assumption that there exist two distinct solutions $v(x, y)$ leads, again by successive approximations, to a contradiction in the usual way.

7. The reciprocity relation of the Green functions. Under assumption (i) [which is identical with assumption (i*)] choose φ, ψ and ρ, σ as follows:

$$\varphi(x) = \exp \int_x^\xi -b(s, \eta)\, ds, \qquad \psi(y) = \exp \int_y^\eta -a(\xi, t)\, dt \tag{5}$$

and

$$\rho(x) = \exp \int_x^\xi b(s, \eta)\, ds, \qquad \sigma(y) = \exp \int_y^\eta a(\xi, t)\, dt \tag{5 bis}$$

(so that $\rho = 1/\varphi$, $\sigma = 1/\psi$). Then conditions (ii), (3) and (ii*), (3 *bis*) are satisfied. Hence both of the above theorems (those of Sec. 2 and Sec. 5) are applicable if a, b, c are just continuous. With reference to the rectangle Q, having $(0, 0)$ and (ξ, η) as opposite vertices, let $G(x, y) = G(x, y; \xi, \eta)$ denote that solution $u(x, y)$, subject to condition (4), of Eq. (1) which belongs to the Cauchy data (5), and let $H(x, y) = H(x, y; \xi, \eta)$ denote that solution $v(x, y)$, subject to condition (4*), of Eq. (1*) which belongs to the Cauchy data (5 *bis*). Then, if (x', y'), (x'', y'') is any pair of points contained in the rectangle Q, Riemann's reciprocity theorem

$$H(x', y'; x'', y'') = G(x'', y''; x', y') \tag{6}$$

is valid.

Since a, b (and c) are now just continuous, the classical proof of the symmetry relation[5], a proof based on Green's formula, fails to apply. One can however conclude as follows.

It is readily seen that the successive approximations leading to $u = G(x', y'; x'', y'')$ and $v = H(x', y'; x'', y'')$ are uniform in the points (x', y'), (x'', y'') of Q and in the index $k(= 1, 2, \cdots)$ together, if (a_1, b_1, c_1), (a_2, b_2, c_2), $\cdots$, (a_k, b_k, c_k), $\cdots$ are sequences of functions which are continuous on Q and tend to (a, b, c) uniformly on Q. Hence, the relation (6) is true for (a, b, c) if it is true for every (a_k, b_k, c_k). Since it is true in the classical case, it follows for the case of just continuous coefficient functions (a, b, c); in fact, $a_k(x, y)$, $b_k(x, y)$, $c_k(x, y)$ can be chosen to be polynomials.

8. The extended validity of Riemann's formula. Without any reference to the relation (6) (which will not be used directly), consider again the problem of Sec. 2, represented by Eqs. (1), (2) and (3), with (i) and (ii) as assumptions. According to Picard's theorem, this problem has on Q a unique solution,

$$u = u(x, y; a, b, c, \varphi, \psi) \tag{7}$$

satisfying condition (4). Riemann's classical formula represents this unique solution (7) in terms of the Green function $H(x, y; \xi, \eta)$ of the adjoint equation. But the classical definition of H is meaningless under the present assumptions (see Sec. 3). On the other hand, if the continuous function $H(x, y; \xi, \eta)$ is defined, not in the traditional manner, but in the way specified in Sec. 7 (as supplied by the existence and uniqueness theorem

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[5]See, e.g., G. Darboux, *op. cit.*, p. 81

of Sec. 6), then *Riemann's integral representation of the solution* (7) *remains valid under the assumptions*, (i) *and* (ii), *of Picard's theorem alone.*

In fact, Riemann's integral representation[6] of the solution (7) contains only the following elements: (α) the boundary data $\varphi(x)$, $\psi(y)$ [which are subject to condition (3)] and their first derivatives and (β) the function $H(x, y; \xi, \eta)$ and their *first* derivatives on the boundary; cf. (5), (5 *bis*). But item (β) does not involve the *second* derivatives of the function H (derivatives which do in general exist). On the other hand, not only the functions $\varphi(x)$, $\psi(y)$ but also their first derivatives, introduced by item (α), are controlled by assumption (ii). Hence it is clear that the italicized assertion, concerning the general validity of Riemann's formula, can be concluded by the same argument (this time by approximating *$a(x, y)$, $b(x, y)$ and $\varphi(x)$, $\psi(y)$ as well as $d\varphi(x)/dx$, $d\psi(y)/dy$* by sequences of smooth functions) which was applied in Sec. 7.

NOTE ON THE AERODYNAMIC HEATING OF AN OSCILLATING INSULATED SURFACE*

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The effect of disturbing the thermal equilibrium of an oscillating conducting surface and its surroundings by changing the isothermal surface temperature at a given time was investigated in Ref. [1]. It was shown therein that the heat transfer and the thermal state of the fluid associated with the oscillating surface can be significantly different from that for conduction from a stationary surface with the same initial temperature difference. To complete this study it is appropriate to investigate the effect of insulating the surface at a given time on the equilibrium state.

Accordingly, consideration is given herein to a doubly infinite plane surface which is oscillating axially (i.e., longitudinally) in a viscous and heat-conducting fluid. It is assumed that sufficient time has elapsed so that an equilibrium state exists in which a periodic motion of the fluid has been established, and the heat obtained by viscous dissipation is all conducted through the surface so that the temperature does not increase indefinitely with time. In this state the fluid velocity is given by [2]

$$u(y, t) = U \exp [-(n/2\nu)^{1/2}y] \cos [nt - (n/2\nu)^{1/2}y] \tag{1}$$

and the temperature is [1]

$$T_e = T_\infty - \frac{U^2 Pr}{4c_p}\left\{\exp [-(2n/\nu)^{1/2}y] - \frac{1}{2 - Pr} [\exp \{-(n/\alpha)^{1/2}y\} \cos \{2nt - (n/\alpha)^{1/2}y\} - \exp \{-(2n/\nu)^{1/2}y\} \cos \{2nt - (2n/\nu)^{1/2}y\}]\right\}, \tag{2}$$

where u is the fluid velocity component parallel to the surface, y is the coordinate normal to the surface, t denotes time, U and n are the amplitude and frequency of the surface

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[6]See, e.g., G. Darboux, *op. cit.*, formula (16) on p. 80

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oscillations, ν is the kinematic viscosity coefficient, Pr is the Prandtl number, c_p is the specific heat at constant pressure, T denotes temperature, α is the thermal diffusivity, and the subscripts e and ∞ denote conditions at equilibrium and far from the surface, respectively.

From Eq. (2) it can be seen that the thermal boundary condition at the surface ($y = 0$) for equilibrium is

$$T_e(0, t) \equiv T_{ew} = T_\infty - \frac{U^2 Pr}{4c_p} . \tag{3}$$

In the problem treated herein the equilibrium is to be disturbed by replacing, at time t_0, the isothermal condition specified by Eq. (3) by one for an insulated surface

$$T_y(0, t) = 0 \qquad t > t_0 , \tag{4}$$

where the subscript denotes partial differentiation. The other boundary conditions are

$$T(\infty, t) = T_\infty \tag{5}$$

$$T(y, t_0) = T_e(y, t_0) \tag{6}$$

and the corresponding differential equation is:

$$T_t - \alpha T_{yy} = \frac{\nu}{c_p} (u_y)^2. \tag{7}$$

Using Laplace transforms, the solution of the problem defined by Eqs. (4) to (7) is found to be:

$$\begin{aligned} T(y, t) = T_e(y, t) &+ \frac{U^2 Pr}{2c_p} \left(\frac{n}{2\nu}\right)^{1/2} \{ -y \, erfc(y/2[\alpha(t - t_0)]^{1/2}) \\ &+ 2[\alpha(t - t_0)/\pi]^{1/2} \exp [-y^2/4\alpha(t - t_0)]\} \\ &+ \frac{U^2}{2c_p} \frac{(n/2\pi)^{1/2}}{1 + (2/Pr)^{1/2}} \int_{t_0}^{t} \cos (2n\tau + \pi/4) \exp [-y^2/4\alpha(t - \tau)](t - \tau)^{-1/2} \, d\tau. \end{aligned} \tag{8}$$

The profiles associated with the oscillations are shown in Fig. 1 for two special cases for air. The equilibrium profile (I) is one in which the temperature monotonically increases

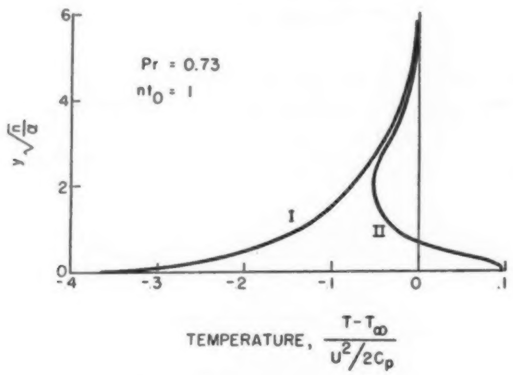

FIG. 1. Temperature profiles.

from the wall to free stream. When the wall is insulated, the wall temperature rises [roughly as $(t - t_0)^{1/2}$] giving a profile (II) which has a temperature minimum in the boundary layer. For larger times the temperature minimum will decrease in magnitude and, presumably move toward the outer edge of the boundary layer. Further, as can be seen from Eq. (8), the thickness of the temperature boundary layer increases linearly with $(t - t_0)$.

Although this solution was developed for an incompressible viscous fluid with constant property values, it is equally valid in the case of a compressible viscous fluid if the boundary-layer assumptions are made, if the Prandtl number and the product of the density, ρ, and absolute viscosity coefficients are assumed constant and if y is replaced by η where

$$\eta = \int_0^y \frac{\rho}{\rho_\infty}\, d\xi.$$

Under these assumptions the compressible boundary-layer equations reduce to those for an incompressible fluid.

The most important property of an insulated surface is perhaps the recovery factor. Using Eqs. (3) and (8), this is

$$r \equiv \frac{T(0, t) - T_\infty}{U^2/2c_p} = -\frac{Pr}{2} + [2Pr\, n(t - t_0)/\pi]^{1/2} + \frac{1}{(\pi)^{1/2}[1 + (2/Pr)^{1/2}]} \int_0^{[2n(t-t_0)]^{1/2}} \cos(2nt + \pi/4 - \beta^2)\, d\beta.$$

Since the time dependence itself is not of primary importance, the recovery factor is averaged over a cycle to yield

$$\begin{aligned}\bar{r} \equiv \frac{n}{\pi}\int_k^{k+\pi/n} r\, dt = \frac{-Pr}{2} &+ \frac{2}{3}(2Pr)^{1/2}\{[1 + n(k - t_0)/\pi]^{3/2} - [n(k - t_0)/\pi]^{3/2}\} \\ &- \frac{\sin(2nt_0 + \pi/4)}{\pi(2)^{1/2}[1 + (2/Pr)^{1/2}]}\{[1 + n(k - t_0)/\pi]^{1/2} - [n(k - t_0)/\pi]^{1/2}\} \\ &+ \frac{1}{2\pi^{3/2}[1 + (2/Pr)^{1/2}]}\int_{[2n(k-t_0)]^{1/2}}^{[2n(k-t_0)+2\pi]^{1/2}} \sin(2nk + \pi/4 - \beta^2)\, d\beta.\end{aligned}$$

Comparing this result with that for a semi-infinite plate in steady flow (for which $r_s = Pr^{1/2}$) and writing the integral in terms of Fresnel integrals yields

$$\begin{aligned}r_1 \equiv \frac{\bar{r}}{r_s} = -\frac{1}{2}(Pr)^{1/2} &+ \frac{(2)^{3/2}}{3}\{[1 + n(k - t_0)/\pi]^{3/2} - [n(k - t_0)/\pi]^{3/2}\} \\ &- \frac{\sin(2nt_0 + \pi/4)}{2\pi[1 + (Pr/2)^{1/2}]}\{[1 + n(k - t_0)/\pi]^{1/2} - [n(k - t_0)/\pi]^{1/2}\} \\ &+ \frac{1}{4\pi[1 + (Pr/2)^{1/2}]}\left\{\sin(2nk + \pi/4)\int_{2[n(k-t_0)/\pi]^{1/2}}^{2[1+n(k-t_0)/\pi]^{1/2}} \cos\frac{\pi}{2}\beta^2\, d\beta\right. \\ &\left. - \cos(2nk + \pi/4)\int_{2[n(k-t_0)/\pi]^{1/2}}^{2[1+n(k-t_0)/\pi]^{1/2}} \sin \pi\beta^2/2\, d\beta\right\}. \qquad (9)\end{aligned}$$

Obviously, the recovery factor with oscillations is increased ($r_1 > 1$) or decreased ($r_1 < 1$) with respect to that for steady flows depending on the relative orders of magnitude of the various terms in Eq. (9). Let us consider the first cycle (i.e., $k = t_0$). Then Eq. (9) can be written, after some simplification, as

$$r_1 = \left[\frac{2^{3/2}}{3} - \frac{1}{2}(Pr)^{1/2}\right] - \frac{0.1235}{1 + (Pr/2)^{1/2}} \sin(2nt_0 - 0.2233). \tag{10}$$

On the other hand, if $n(k - t_0)/\pi$ is appreciable (over 1/2, say) then the first two terms of Eq. (9) give r_1 with an error of less than 5 per cent for air.

For air, ($Pr = 0.73$) Eq. (10) shows that, for $k = 0$,

$$0.439 \leq r_1 \leq 0.593,$$

depending on the value of nt_0. On the other hand, for large k, the recovery factor increases as $(k - t_0)^{1/2}$, which increase is directly due to the energy added to the boundary layer by the plate oscillations. This variation is shown in Fig. 2. It is seen that advantageous recovery factors (i.e., $r_1 < 1$) will only persist for a short time, until the initially cold boundary layer (Fig. 1, I) is heated by the plate oscillations.

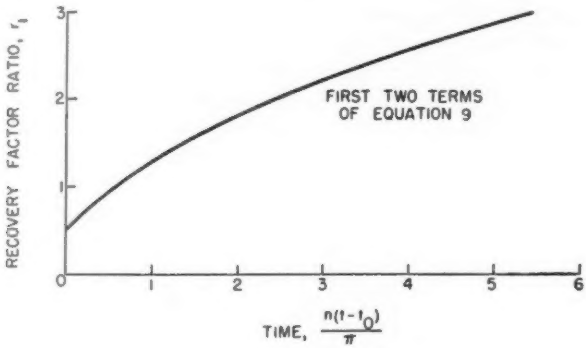

FIG. 2. Recovery factor ratio.

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A UNIQUENESS THEOREM FOR THE COUPLED THERMOELASTIC PROBLEM*

By J. H. WEINER (*Columbia University*)

1. Introduction. In the computation of thermal stresses in an elastic solid it is customary to compute first the temperature distribution by use of the Fourier heat conduction equation and then to determine the resulting thermal stresses according to the usual thermoelastic theory. Although this procedure is sufficiently accurate for a large class of problems, it is approximate since the Fourier heat conduction equation is an energy balance which neglects the interconvertibility of mechanical and thermal energy. If this possibility is included in the analysis, the energy balance equation contains both thermal and mechanical terms and the thermal and thermoelastic problems are coupled; the temperature and stress distributions must be determined simultaneously rather than consecutively. When the non-uniform temperature distribution is primarily due to heat supplied to the body from external sources, the mechanical coupling term in the energy balance may be neglected in comparison with the thermal terms; when the temperature differences are due solely to the deformations of the body, as in a study of thermoelastic damping for example, then the coupled problem must be considered.

The coupled nature of the thermal-thermoelastic problem was already known by Duhamel [1]; a derivation of the governing equations utilizing thermodynamic principles[1] was given by Voigt [2] who presents a linear theory valid for sufficiently small temperature changes, displacements and displacement gradients. The magnitude of permissible temperature changes depends upon the mean absolute temperature of the solid and the degree of temperature dependence of its elastic and thermal properties, while the restrictions on the displacements and their gradients are those required in the linear theory of elasticity.

In this note, a uniqueness theorem is presented for the coupled problem formulated by Voigt for the case of an isotropic elastic solid. The notation used is as follows: σ_{ij}, e_{ij}, u_i are components of the stress, strain and displacement tensors referred to a cartesian coordinate system $x_i(i, j = 1, 2, 3)$, λ, μ are Lamé's constants, ρ is the density, α is the coefficient of thermal expansion and $m = (3\lambda + 2\mu)\,\alpha$. T is the absolute temperature, T_0 is a reference temperature chosen so that $|\,(T - T_0)/T_0\,| \ll 1$ throughout the body, K is the thermal conductivity, c is the specific heat for processes with invariant strain tensor. The comma notation is used for derivatives with respect to space variables, superposed dots for derivatives with respect to the time, t, δ_{ij} is the Kronecker delta and the summation convention is employed.

2. Theorem. Given a regular region[2] of space $V + S$ with boundary S. Then there exists at most one set of single-valued functions $\sigma_{ij}(P, t)$ and $e_{ij}(P, t)$ of class $C^{(1)}$, $u_i(P, t)$ and $T(P, t)$ of class $C^{(2)}$ for $P(x_1, x_2, x_3)$ in $V + S$, $t \geq 0$ which satisfy the following equations for P in V, $t > 0$,

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[1]It should be noted here that these thermodynamic principles, as applied to deformable media, have not yet been put on a firm logical foundation, see [3], pp. 170–171. See also [7], which came to the author's attention after submission of the manuscript, for a comprehensive treatment of this subject from the viewpoint of irreversible thermodynamics and a physical interpretation of the integral obtained in Eq. (15) below.

[2]As defined in [4], p. 113.

$$KT_{,kk} = \rho c T^{\cdot} + mT_0 e^{\cdot}_{kk}, \tag{1}$$

$$\sigma_{ij,j} = \rho u^{\cdot\cdot}_i, \tag{2}$$

the following equations for P in $V + S$, $t \geq 0$,

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{3}$$

$$\sigma_{ij} = \delta_{ij}\lambda e_{kk} + 2\mu e_{ij} - \delta_{ij} mT, \tag{4}$$

the following equations for P on S, $t > 0$,

$$T = F^{(1)}(P, t), \tag{5}$$

$$u_i = G_i^{(1)}(P, t), \tag{6}$$

and the following equations for P in V, $t = 0$

$$T = F^{(2)}(P), \tag{7}$$

$$u_i = G_i^{(2)}(P), \tag{8}$$

$$u^{\cdot}_i = G_i^{(3)}(P), \tag{9}$$

where the constants K, c, λ, μ, m and T_0 are all positive.

Proof. Let there be two such sets of functions, $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$, $e_{ij}^{(1)}$ and $e_{ij}^{(2)}$ etc., and let $\sigma_{ij}^* = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$, $e_{ij}^* = e_{ij}^{(1)} - e_{ij}^{(2)}$, etc. By virtue of the linearity of the problem, it is clear that these difference functions will also satisfy Eqs. (1)-(4) and the homogeneous counterparts of Eqs. (5)-(9). In the calculations which follow the stars will be omitted from the designations of the difference functions. Consider the integral

$$\int_V \sigma_{ij} e^{\cdot}_{ij}\, dV = \int_V \sigma_{ij} u^{\cdot}_{i,j}\, dV = \int_V [(\sigma_{ij} u^{\cdot}_i)_{,j} - (\sigma_{ij,j} u^{\cdot}_i)]\, dV, \tag{10}$$

where Eq. (3) and the symmetry of the stress tensor have been utilized. By use of the divergence theorem and the homogeneous form of Eq. (6),

$$\int_V (\sigma_{ij} u^{\cdot}_i)_{,j}\, dV = \int_S \sigma_{ij} u^{\cdot}_i n_j\, dS = 0.$$

Also, from Eq. (2),

$$\int_V \sigma_{ij,j} u^{\cdot}_i\, dV = \int_V \rho u^{\cdot\cdot}_i u^{\cdot}_i\, dV = \int_V \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^{\cdot}_i u^{\cdot}_i\right) dV.$$

Therefore, Eq. (10) becomes,

$$\int_V \left[\sigma_{ij} e^{\cdot}_{ij} + \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^{\cdot}_i u^{\cdot}_i\right)\right] dV = 0. \tag{11}$$

From Eq. (4) it is found that

$$\sigma_{ij} e^{\cdot}_{ij} = \lambda e_{mm} e^{\cdot}_{kk} + 2\mu e_{ij} e^{\cdot}_{ij} - mT e^{\cdot}_{kk} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\lambda e_{kk}^2 + \mu e_{ij} e_{ij}\right) - mT e^{\cdot}_{kk}$$

so that Eq. (11) may be rewritten in the form

$$\int_V \left[\frac{\partial}{\partial t}\left(\tfrac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \tfrac{1}{2}\rho u_i^{\cdot 2}\right) - mT\dot{e}_{kk}\right] dV = 0. \tag{12}$$

The following identity is readily derived by use of the divergence theorem

$$\int_V TT_{,kk}\, dV + \int_V T_{,k}T_{,k}\, dV = \int_V (TT_{,k})_{,k}\, dV = \int_S TT_{,k}n_k\, dS.$$

Substitution for $T_{,kk}$ from Eq. (1) in the above and use of the homogeneous form of Eq. (5) yields

$$\int_V T(\rho c T^{\cdot} + mT_0\dot{e}_{kk})\, dV + K\int_V T_{,k}T_{,k}\, dV = 0$$

or

$$T_0 \int_V mT\dot{e}_{kk}\, dV = \frac{-\rho c}{2}\int_V \frac{\partial}{\partial t} T^2\, dV - K\int_V T_{,k}T_{,k}\, dV. \tag{13}$$

Substitution of Eq. (13) into Eq. (12) and interchange of the order of differentiation and integration then yields,

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \frac{1}{2}\rho u_i^{\cdot 2} + \frac{\rho c}{2T_0}T^2\right) dV = \frac{-K}{T_0}\int_V T_{,k}T_{,k}\, dV \leq 0. \tag{14}$$

The integral on the left hand side of the above equation is initially zero since the difference functions satisfy homogeneous initial conditions. By the inequality derived above, however, this integral either decreases (and therefore becomes negative) or remains equal to zero. Since its integral is the sum of squares, however, only the latter alternative is possible. That is

$$\int_V \left(\frac{1}{2}\lambda e_{kk}^2 + \mu e_{ij}e_{ij} + \frac{1}{2}\rho u_i^{\cdot 2} + \frac{\rho c}{2T_0}T^2\right) dV = 0, \qquad t \geq 0. \tag{15}$$

It follows from Eq. (15) that the difference functions are identically zero throughout the body and for all time and the theorem is proved.

3. Remarks. As noted in the Introduction, omission of the last term in Eq. (1) reduces the problem defined by Eqs. (1)-(9) to a heat conduction problem and a thermoelastic problem which are uncoupled. In this case, the right hand side of Eq. (13) is equal to zero and this equation may be used directly to prove the uniqueness of the temperature distribution. It follows then that the difference temperature distribution, T, in Eq. (12) is zero and that equation may be used in the usual manner to prove the uniqueness of the solution of the thermoelastic problem.

The above theorem may be generalized readily to include the most general linear thermal and mechanical boundary conditions[3]. Also the analytical restrictions on the solutions may be considerably lightened. These generalizations have not been considered here since the primary purpose of this note is to indicate how methods of proving uniqueness of the uncoupled heat conduction and elastic problems may be combined

[3]More general mechanical boundary conditions which are sufficient to prove uniqueness are, in fact, implicit in the second integral of the equation following Eq. (10).

to prove uniqueness of the coupled problem. The reader is therefore referred to other sources, e.g., [5, 6], for more detailed treatments of the uncoupled problems.

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THE MOTION OF A THERMOELASTIC SOLID*

By M. LESSEN (*University of Pennsylvania*)

Introduction. It is the purpose of this paper to set down formally the relevant equations of thermo-elasticity within the approximation of the theory of elasticity of infinitesimal displacements and displacement derivatives. It is not claimed that the following equations of thermo-elasticity are original; Duhamel [1] and Neumann [2] derived similar equations many years ago, but due to the fact that they did not derive their equations from thermodynamic considerations and also that elasticians generally are not aware of the role of thermodynamics in their field, it was felt that a derivation of the general equations and application to a particular problem were in order. The present work is a refinement of Refs. [3] and [4] and the application was inspired by the work of Synge [5] in connection with the motion of a viscous, heat conducting fluid. The author is indebted to the reviewer for his extensive commentary which assisted materially in the revised version of this paper.

Analysis. For the case of small displacements and displacement derivatives, the momentum and energy (First Law) equations for a continuous, homogeneous medium may be written as

$$\text{Momentum} \qquad \rho \frac{\partial^2 u_i}{\partial t^2} = \tau_{ki,k} \, , \tag{1}$$

$$\text{Energy} \qquad \rho \frac{\partial U}{\partial t} = K_{mn} T_{,mn} + \tau_{mn} \frac{\partial u_{n,m}}{\partial t} \, , \tag{2}$$

where ρ is density, u_i is displacement vector, t is time coordinate, τ_{ki} is stress tensor, U is specific internal energy, K_{mn} is thermal conductivity tensor, and T is the temperature. The subscript notation is that of cartesian tensorial form and subscripts following a

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comma indicate partial differentiation with respect to the appropriate independent spatial variables.

The momentum and energy equations constitute four equations connecting the 11 dependent variables:

$$u_i \ ; \tau_{ij} \ ; U; T$$

therefore, seven additional relations must be obtained, in order to reduce the number of dependent variables to four. This reduction can be accomplished thermodynamically.

For an elastic solid, where τ_{ij}; ϵ_{ij}; U; and T are thermodynamic properties of a state, the Gibbs total differential equation may be written as

$$T\,dS = dU - \frac{1}{\rho}\,\tau_{ij}\,d\epsilon_{ij}\ , \tag{3}$$

where S is the specific entropy, and $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is equal to the "pure" strain. Therefore, from an assumed equation of state of the (normal) form

$$U = U(S, \epsilon_{ij}), \tag{4}$$

T and τ_{ij} can be obtained at once by

$$T = \frac{\partial U}{\partial S}\ ; \qquad \tau_{ij} = \rho\,\frac{\partial U}{\partial \epsilon_{ij}}. \tag{5}$$

Expanding (4) about the reference state $S = S^{(0)}$; $\epsilon_{ij} = 0$, including second order terms, we obtain

$$U = U^{(0)} + sU^{(1)} + A_{ij}\epsilon_{ij} + \frac{1}{2}\,s^2U^{(2)} + sB_{ij}\epsilon_{ij} + \frac{1}{2\rho}\,C_{ijmn}\epsilon_{ij}\epsilon_{mn}\ , \tag{6}$$

where $s = S - S^{(0)}$; $U^{(1)}$, $U^{(2)}$, A_{ij} , B_{ij} , C_{ijmn} are the known properties $\partial U/\partial s$, $\partial^2 U/\partial s^2$, $\partial U/\partial \epsilon_{ij}$, $\partial^2 U/\partial\epsilon_{ij}\partial s$, $\rho\partial^2 U/\partial\epsilon_{ij}\partial\epsilon_{mn}$ respectively at the reference state. Therefore

$$T = U^{(1)} + sU^{(2)} + B_{ij}\epsilon_{ij}\ , \tag{7}$$

$$\tau_{ij} = \rho A_{ij} + \rho s B_{ij} + C_{ijmn}\epsilon_{mn}\ . \tag{8}$$

$U^{(1)}$ is seen in (7) to be the reference temperature $T^{(0)}$.

Substituting (7) and (8) in (1) and (2) yields (to the first order)

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} = \rho B_{ij} s_{,j} + C_{ijmn} u_{m,nj} \tag{9}$$

and

$$\rho T^{(0)}\,\frac{\partial s}{\partial t} = K_{mn}(sU^{(2)} + B_{ij}u_{i,j})_{,mn}\ . \tag{10}$$

Thus, the reduction to four variables, s and u_i has been accomplished.

The reduction to the variables T and u_i may be accomplished by solving for s in (7),

$$s = \frac{1}{U^{(2)}}\,(T - T^{(0)} - B_{mn}u_{m,n}) \tag{11}$$

and substitution in (9) and (10)

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{U^{(2)}} B_{ij} T_{,j} + \left(\frac{1}{\rho} C_{ijmn} - \frac{1}{U^{(2)}} B_{ij} B_{mn}\right) u_{m,nj} , \tag{12}$$

$$\frac{\partial T}{\partial t} - B_{ij} \frac{\partial u_{i,j}}{\partial t} = D_{mn} T_{,mn} , \tag{13}$$

where

$$D_{mn} = \frac{U^{(2)}}{\rho T^{(0)}} K_{mn} . \tag{14}$$

For the case of isotropy,

$$B_{ij} = B\delta_{ij} ; \qquad D_{mn} = D\delta_{mn} ,$$

$$C_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$$

and (12) and (13) become

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{B}{U^{(2)}} T_{,i} + \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}}\right) u_{m,mi} + \frac{\mu}{\rho} u_{i,mm} , \tag{15}$$

$$\frac{\partial T}{\partial t} - B \frac{\partial u_{m,m}}{\partial t} = DT_{,mm} . \tag{16}$$

It should be noted that Eq. (16) resembles the customary heat conduction equation except for the term $B\partial u_{m,m}/\partial t$. Duhamel included a similar term in his considerations, but did so because he reasoned that the rate of dilatation would have a linear effect on the rate of temperature change. While it is true that the effect of B is small for many elasticity problems, many elasticians are unaware of the nature of the approximation they make when they use the Fourier equation to obtain a space-time temperature distribution.

Propagation of waves. In the manner of Synge, let us now study solutions of Eqs. (15) and (16) of the form

$$u_i = u_i^* \exp\,(a_k x_k + bt),$$

$$T - T^{(0)} = T^* \exp\,(a_k x_k + bt),$$

where a_i , b are complex.
Substituting in Eqs. (15) and (16) and letting $a_k a_k = A^2$; one obtains

$$\left(b^2 - \frac{\mu}{\rho}\right) A^2 u_i^* - \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}}\right) a_m a_i u_m^* - \frac{B}{U^{(2)}} a_i T^* = 0, \tag{17}$$

$$-Bba_m u_m^* + (b - DA^2)T^* = 0. \tag{18}$$

If $a_i = a_i' + ia_i''$ where a_i' , a_i'' are real, then if we choose the x_3 axis perpendicular to the vectors a_i' and a_i'' , $a_3 = 0$. The determinantal equation for the system then is

$$\begin{vmatrix} \left[b^2 - \frac{\mu}{\rho} A^2 - \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}}\right) a_1^2\right] & -\left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}}\right) a_1 a_2 & -\frac{B}{U^{(2)}} a_1 \\ -\left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}}\right) a_1 a_2 & \left[b^2 - \frac{\mu}{\rho} A^2 - \left(\frac{\lambda + \mu}{\rho} - \frac{B^2}{U^{(2)}}\right) a_2^2\right] & -\frac{B}{U^{(2)}} a_2 \\ -Bba_1 & -Bba_2 & (b - DA^2) \end{vmatrix} = 0 \tag{19}$$

or, letting

$$\frac{\mu}{\rho} = V^2; \qquad \frac{\lambda + 2\mu}{\rho} \qquad -\frac{B^2}{U^{(2)}} = V_T^2$$

and

$$\frac{\lambda + 2\mu}{\rho} = V_S^2 ,$$

the determinantal equation may be written

$$(b^2 - V^2A^2)[(b^2 - V_S^2A^2)b - (b^2 - V_T^2A^2)DA^2] = 0. \tag{20}$$

V, V_s and V_T are clearly velocities of propagation of transverse, isentropic-longitudinal, and isothermal-longitudinal disturbances, respectively. The transverse mode is uncoupled from the longitudinal mode of motion.

A few limiting cases are of interest. For $B \to 0$; $V_s \to V_T$ and the mechanical mode uncouples from the temperature mode. The propagation of a temperature disturbance is then given by

$$b - DA^2 = 0. \tag{21}$$

For $D \to 0$; a longitudinal disturbance is given by

$$b^2 - V_S^2A^2 = 0. \tag{22}$$

For $D \to \infty$; a longitudinal disturbance is given by

$$b^2 - V_T^2A^2 = 0. \tag{23}$$

The general equation for a longitudinal disturbance is seen, from Eq. (20), to be biquadratic in A. The two pairs of roots, therefore, represent advancing and receding waves of two families, each family having a different velocity of propagation. It is, therefore, observed that the herein studied continuum model displays a "second sound" phenomenon and that disturbances of both families exhibit frequency dependent damping and velocities of propagation.

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A NOTE ON POISSON'S INTEGRAL*

BY R. J. DUFFIN (*Carnegie Institute of Technology*)

Let D be a convex open set with boundary B. Let f be a function defined at the points of B and continuous at these points. This note concerns a mean value formula which extends f to be a continuous function in $D + B$. At first discussion is restricted to the case that D is a bounded set either in space or in the plane.

A line drawn through a point p of D will intersect B in exactly two points, say q_1 and q_2, as is illustrated in Fig. 1. The value of the function f at these points will be

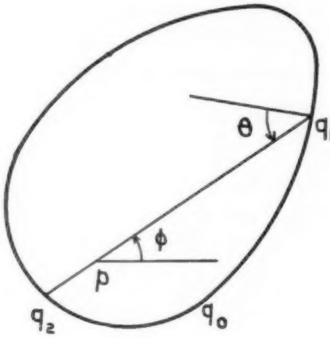

FIG. 1.

denoted by f_1 and f_2. A linear function g is defined on this line so that g takes on the value f_1 at q_1 and the value f_2 at q_2. The value of g at a given point p is seen to be a continuous function of p and of the direction of the line through p. With p held fixed, let $F(p)$ denote the average of g for all directions of the line. Thus $F(p)$ is a continuous function for p in D. (It is of interest to note that if f is the boundary value of a linear function h, then $F \equiv h$ in D.)

Let q_0 be a point on B where B is strictly convex. It is desired to show that if $p \to q_0$, then $g \to f_0$ uniformly with respect to the direction of the line through p. If this is not true, there is a sequence of points p_n such that $p_n \to q_0$ and $| g - f_0 | > c_1$ for a positive constant c_1. By strict convexity $| q_1 - q_0 || q_2 - q_0 | \to 0$. Without loss of generality it may be assumed that $q_1 \to q_0$. If there is a positive constant c_2 such that $| q_2 - q_1 | > c_2$, then by continuity and the definition of g, it is seen that $g \to f_0$. If there is no such c_2, then for some infinite subsequence both $q_1 \to q_0$ and $q_2 \to q_0$ and so $g \to f_0$. In either case there is a contradiction, and so g must converge uniformly to f_0. It follows immediately that $F(p) \to f(q_0)$ as $p \to q_0$. This statement is seen to hold at an arbitrary point of the boundary by slightly modifying the above proof.

Let r_1 be the distance from p to q_1 and let r_2 be the distance from p to q_2. Thus

$$g = (r_1 f_2 + r_2 f_1)/(r_1 + r_2). \tag{1}$$

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In the two-dimensional case

$$F(p) = \frac{1}{2\pi} \int_0^{2\pi} g\, d\phi = \frac{1}{\pi} \int_0^{2\pi} \frac{r_2 f_1}{r_1 + r_2}\, d\phi. \tag{2}$$

Here ϕ denotes the angle the line pq_1 makes with a fixed line. If the boundary is sufficiently smooth, (2) can be transformed into a line integral. Let θ be the angle formed by the normal to B at q_1 and the line pq_1. Then $r_1\, d\phi = \cos\theta\, ds$ where ds denotes the element of arc length of B. Thus

$$F(p) = \frac{1}{\pi} \int_0^c \frac{\cos\theta}{r_1} f_1\, ds - \frac{1}{\pi} \int_0^c \frac{\cos\theta}{r_1 + r_2} f_1\, ds, \tag{3}$$

where c is the length of B. It is clear that the first integral is harmonic in D. In the case that B is a circle of radius a, it is apparent that $r_1 + r_2 = 2a \cos\theta$. Thus the second integral is constant and so F is harmonic if B is a circle.

In three dimensions formulas (2) and (3) become

$$F(p) = \frac{1}{2\pi} \iint \frac{r_2 f_1}{r_1 + r_2}\, d\Omega \tag{2'}$$

and

$$F(p) = \frac{1}{2\pi} \iint \frac{\cos\theta}{r_1^2} f_1\, dS - \frac{1}{2\pi} \iint \frac{\cos\theta}{(r_1 + r_2)} \frac{f_1}{r_1}\, dS. \tag{3'}$$

Here $d\Omega$ is the element of solid angle and dS is the element of surface area. The first integral in (3′) is seen to be harmonic. If B is a sphere of radius a, then again $r_1 + r_2 = 2a \cos\theta$ and it is seen that the second integral in (3′) is harmonic.

Thus in the case of the circle and the sphere the mean value formula yields a harmonic function. Since the Dirichlet problem has a unique solution, it follows that the mean value formula is equivalent to Poisson's integral. The formula may also be regarded as an extension of the Gauss mean value relation. There is no difficulty in extending the mean value formula to hold for infinite regions. This yields relations equivalent to Poisson's formula for the half-plane and the half-space.

For regions of the plane not too far from circular shape it is to be expected that the mean value formula (2) would give an approximate solution of the Dirichlet problem. Furthermore, the integral is suitable for approximation by a discrete sum. This is the method of *the linear rosette* proposed by M. M. Frocht in his book *Photoelasticity*, vol. II, John Wiley, New York, 1948. Frocht carries out some examples of the approximation method and makes comparison with the exact solution.

Again consider a region of the plane. An analytic function is defined by

$$W(p) = \frac{1}{\pi} \int_0^c \frac{e^{-i\theta}}{r_1} f_1\, ds. \tag{4}$$

Let $W = U + iV$ then

$$U = \frac{1}{\pi} \int_0^{2\pi} f_1\, d\phi \tag{5}$$

and

$$V = -\frac{1}{\pi} \int_0^{2\pi} \tan\theta f_1\, d\phi. \tag{6}$$

It is seen that U is identical with the first integral in (3). Thus for a circle, U differs from F by a constant, and so V is the harmonic conjugate of F.

For regions not too far from circular it is to be expected that $U + C$ where C is a constant would approximate the solution of the Dirichlet problem. Of course $U + C$ would not take on the correct boundary values. By the maximum principle it results that the greatest error would be on the boundary; hence the error could be determined.

Since the error in the above procedure is harmonic, a second approximation could be set up, etc. This would lead to the *method of the arithmetic mean* devised by Neumann to solve the Dirichlet problem in convex regions.

ON SOME EFFECTS OF VELOCITY DISTRIBUTION IN ELECTRON STREAMS

Quarterly of Applied Mathematics, XII, 105-116 (1954)

By S. V. YADAVALLI

Although the details of the derivation are different Equations (83), (73) and (67) derived earlier by J. K. Knipp (see: "Klystrons and Microwave Triodes", M.I.T. Radiation Laboratory Series, Vol. 7, McGraw-Hill, Chapter 5) are somewhat more general than the corresponding Equations (11), (14) and (43a) of my paper. This was not indicated in my above paper due to an oversight. The author would like to thank Professor J. K. Knipp for kindly drawing his attention to this matter.

BOOK REVIEWS

(Continued from p. 82)

Théorie des fonctions aléatoires. Applications a divers phénomènes de fluctuation. By A. Blanc-Lapierre and Robert Fortet. Masson et Cie, Editeurs, Libraires de l'Académie de Médecine, 120 Boulevard Saint-Germain, Paris, 1953. xvi + 693 pp. $18.77.

The book before us sets itself the difficult task to serve the needs both of the mathematician who wants to know the relevant problems and of the scientist who needs and applies the theory. There is always a gap between theory and applications; here, however, the situation is perhaps particularly strained: The domain is difficult and comparatively little known: the authors rightly say that a research worker who may be able and willing to acquaint himself to a certain degree with a branch of analysis new to him will in general experience considerable difficulties and inhibitions if probability enters the field. (If this is valid in France, the homeland of probability calculus, it is certainly more than true in this country.) Random functions appear in the most varied branches of knowledge: in physics, viz. in the theories of turbulence, of Brownian motion, of noise, etc.; in meteorology; in economics in the theory of time series, and in other fields.

The content of the 700-page work is very rich. Some of the chapters speak particularly to the scientist, others to the mathematician. The main topics are: Physical introduction to the theory of random functions (for the scientist); axioms, concepts, and theorems of probability calculus (for the mathematician); random functions; Poisson random process; Laplace random process; Markoff chains; harmonic analysis of random functions; various applications. A comprehensive very valuable chapter by J. Kampé de Fériet deals with the statistical theory of turbulence. The book, on the whole a great and successful effort, has been invited by Professor Darmois for his Series of Mathematical Works for Physicists.

In this reviewer's opinion, the book makes very difficult reading: Much is assumed as known, the presentation is very brief; it is interesting for re-evaluating a subject one knows well, but hard indeed for trying to learn new material. It is a book for the research worker who either wants to get a survey of existing problems and results, or who is willing to give considerable effort to the study of a particular subject, using other books and papers as a complement.

The valuable bibliography contains several hundred titles,—although well known names are missing—; the index, hardly two pages, is meager.

HILDA GEIRINGER

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